Optional Homework

This homework is **OPTIONAL**. It counts for 3% of **BONUS** credit.

Question 1. [20 marks] Prove that if $f : \alpha \to \beta$ is a strictly increasing map between two ordinals, then $f(\gamma) \geq \gamma$ for every $\gamma \in \alpha$. Conclude that if $\alpha \leq \beta$ and f is an order-isomorphism, then $\alpha = \beta$ and f is the identity map.

Question 2. [40 marks] Prove Theorem 1.6.27. More precisely:

(1) Using Question 1 (or otherwise) prove that if such an ordinal exists, then it is unique.

(2) Given an ordered set X and $x \in X$, let $S_{<x} = \{y \in X : y < x\}$. Prove that if there is an isomorphism between $S_{<x}$ and some ordinal α, then there is an isomorphism between $S_{\leq x} = S_{<x} \cup \{x\}$ and α^+.

(3) Consider the set:

$$Y := \{y \in X : S_{\leq y} \text{ is order isomorphic to an ordinal } \alpha\}.$$

and prove that for all $y \in Y$, the ordinal $\alpha(y)$, to which it is isomorphic, and the isomorphism f_y are unique (you can use Part (1) of the question here).

The rest of the problem is about showing that $X = Y$.

(4) Suppose that $Y \neq X$. Show that if $x \in X \setminus Y$ is minimal and $\alpha = \bigcup_{y<x} \alpha(y)$, then there is an order-isomorphism $S_{<x} \to \alpha$ given by $f(y) = f_y(y)$.

(5) Deduce that $X = Y$.

Conclude by defining (similarly to Part (4)) an appropriate ordinal α and order-isomorphism $f : X \to \alpha$.

Question 3. [40 marks] Prove Theorem 1.8.8. More precisely:

(1) Let κ be a cardinal and consider the function:

$$f : \kappa + 1 \to \aleph_{\kappa+1}$$
$$\beta \mapsto \aleph_\beta$$

Using Question 2 (or otherwise) conclude that $\aleph_{\kappa+1} > \kappa$.

(2) Show that if $\alpha \leq \kappa + 1$ is minimal such that $\aleph_\alpha > \kappa$ then $\alpha > 0$.

(3) Show that α cannot be a limit ordinal.

(4) Deduce that there is some ordinal β such that $\aleph_\beta \leq \kappa < \aleph_{\beta^+}$.

Conclude that $\kappa = \aleph_\beta$.