Homework 1

Question 1. [30 marks] Solve the following exercises:

- Exercise 1.4.5
- Exercise 1.4.11
- Exercise 1.5.4

A large part of the course going forward will involve *proofs by induction*. Many of you may have not seen these before, and the following two questions are here to make sure everyone is on the same page.

But first of all, *what* is a "proof by induction"? Well... This is a relatively long story, but here is the gist of it. Often-times we construct a set $\mathcal{I}$ of mathematical objects in the following way: (a) We describe the "basic objects" of $\mathcal{I}$. (b) We describe a way of building larger objects of $\mathcal{I}$ from the objects we have already built. This is called an **inductive** (or **recursive**) definition.

Then, to prove that some property is true of all elements of $\mathcal{I}$, it suffices to prove that: (a) It is true of the "basic objects" of $\mathcal{I}$. (b) If it is true of some objects, then it is true of all larger objects built from them.

That's really abstract, so the next two exercises will try to illustrate it.

Question 2. [30 marks] We can build the set of natural numbers $\mathbb{N}$ by induction as follows: (a) $0 \in \mathbb{N}$ (that's the basic object). (b) If $n \in \mathbb{N}$ then $n+1 \in \mathbb{N}$ (that how to build larger objects from smaller ones). If we then wish to prove that some property $P(n)$ is true of all $n \in \mathbb{N}$ we have to show that: (a) $P(0)$ is true. (0) If $P(n)$ is true, then so is $P(n+1)$. Now, on with the question:

(1) Prove by induction that for all $n \in \mathbb{N}$ the following statement is true:

$$\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

(2) Prove by induction that for all $n \in \mathbb{N}$ the following statement is true:

$$9^n - 1 \text{ is divisible by } 8.$$

If $\mathcal{I}$ is defined by induction, then we can also use induction to define functions on $\mathcal{I}$ – to define a function f on all of $\mathcal{I}$ we just need to define f on the "basic objects" and provided we know how to compute f on simpler objects, we need to define f on larger ones built from them.

We can build more things by induction, not just the natural numbers. This will occupy us a lot in the next chapter, and the next exercise is a sort-of warm up.

Question 3. [40 marks] We define a set $\mathcal{S}$ of strings of symbols, inductively, as follows: (a) $\mathtt{a} \in \mathcal{S}$, and $\mathtt{b} \in \mathcal{S}$ (these are the basic objects). (b) If $s, t \in \mathcal{S}$ then: $(s) \in \mathcal{S}$, $(st) \in \mathcal{S}$ (where st denotes the concatenation of s and t), and $([s] \uparrow [t]) \in \mathcal{S}$ (so in this example we have three ways of building objects from smaller ones). For example:

$$\mathtt{a} \in \mathcal{S}, (\mathtt{a}) \in \mathcal{S}, (\mathtt{aa}) \in \mathcal{S}, ((\mathtt{a})\mathtt{a}) \in \mathcal{S}, ([\mathtt{a}] \uparrow [(\mathtt{ba})]) \in \mathcal{S}.$$

We can inductively define a function $\mathsf{bra} : \mathcal{S} \to \mathbb{N}$ as follows $\mathsf{bra}(\mathtt{a}) := 0$, $\mathsf{bra}(\mathtt{b}) := 0$. Once $\mathsf{bra}(s), \mathsf{bra}(t)$ have been defined, we set:

$$\begin{aligned} \mathsf{bra}((s)) &:= \mathsf{bra}(s) + 2 \\ \mathsf{bra}((st)) &:= \mathsf{bra}(s) + \mathsf{bra}(t) + 2 \\ \mathsf{bra}(([s] \uparrow [t])) &:= \mathsf{bra}(s) + \mathsf{bra}(t) + 6 \end{aligned}$$

(1) Prove that for all $s \in \mathcal{S}$, we have that $\mathsf{bra}(s)$ is even.

Let $x, y \in \mathcal{S}$. We can define a function $\mathsf{Sub}_{[x/\mathtt{a}, y/\mathtt{b}]} : \mathcal{S} \to \mathcal{S}$ as follows $\mathsf{Sub}_{[x/\mathtt{a}, y/\mathtt{b}]}(\mathtt{a}) := x$, $\mathsf{Sub}_{[x/\mathtt{a}, y/\mathtt{b}]}(\mathtt{b}) := y$, and:

$$\begin{aligned} \mathsf{Sub}_{[x/\mathtt{a}, y/\mathtt{b}]}((s)) &:= (\mathsf{Sub}_{[x/\mathtt{a}, y/\mathtt{b}]}(s)) \\ \mathsf{Sub}_{[x/\mathtt{a}, x/\mathtt{b}]}((st)) &:= (\mathsf{Sub}_{[x/\mathtt{a}, y/\mathtt{b}]}(s)\mathsf{Sub}_{[x/\mathtt{a}, y/\mathtt{b}]}(t)) \\ \mathsf{Sub}_{[x/\mathtt{a}, y/\mathtt{b}]}(([s] \uparrow [t])) &:= ([\mathsf{Sub}_{[x/\mathtt{a}, y/\mathtt{b}]}(s)] \uparrow [\mathsf{Sub}_{[x/\mathtt{a}, y/\mathtt{b}]}(t)]) \end{aligned}$$

(2) Prove that for all $s \in \mathcal{S}$, $\mathsf{Sub}_{[\mathtt{a}/\mathtt{a}, \mathtt{a}/\mathtt{b}]}(s)$ does not contain the letter $\mathtt{b}$.

(3) Prove that for all $s \in \mathcal{S}$, $\mathsf{Sub}_{[\mathtt{bb}/\mathtt{a}, \mathtt{aa}/\mathtt{b}]}(s)$ contains an even number of occurrences of each of $\mathtt{a}$ and $\mathtt{b}$.

Finally, define a function $\mathsf{let} : \mathcal{S} \to \mathbb{N}$ as follows: $\mathsf{let}(\mathtt{a}) = 1$, $\mathsf{let}(\mathtt{b}) = 1$, and:

$$\begin{aligned} \mathsf{let}((s)) &:= \mathsf{let}(s) \\ \mathsf{let}((st)) &:= \mathsf{let}(s) + \mathsf{let}(t) \\ \mathsf{let}(([s] \uparrow [t])) &:= \mathsf{let}(s) + \mathsf{let}(t) \end{aligned}$$

(4) Show that for all $s \in \mathcal{S}$, $\mathsf{bra}(\mathsf{Sub}_{[(\mathtt{a})/\mathtt{a}, (\mathtt{b})/\mathtt{b}]}(s)) = \mathsf{bra}(s) + 2\mathsf{let}(s)$.

[*Hint.* I am asking here for a "formal" proof by induction. Clearly write out what you need to prove. This should guide you.]