Homework 3

Question 1. [20 marks] Solve the following exercises from **Week 3(b)** of the notes:

(1) Exercise 1.1.6 [*Hint.* If you think there is an unreasonable number of terms, justify why you don't want to write all of them out.]

(2) Exercise 1.1.9.

(3) Exercise 1.1.11.

Question 2. [20 marks] Let $\mathcal{L}$ be a first-order language with two unary function symbols $\underline{f}$ and $\underline{g}$.

(1) Find three $\mathcal{L}$-sentences ϕ_1, ϕ_2, ϕ_3, such that:

- In every $\mathcal{L}$-structure $\mathcal{M}$, we have that $\mathcal{M} \vDash \phi_1$ if and only if $\underline{f}$ and $\underline{g}$ are constant.
- In every $\mathcal{L}$-structure $\mathcal{M}$, we have that $\mathcal{M} \vDash \phi_2$ if and only if $\mathsf{im}(\underline{f}) \cap \mathsf{im}(\underline{g})$ contains exactly two elements.
- In every $\mathcal{L}$-structure $\mathcal{M}$, we have that $\mathcal{M} \vDash \phi_3$ if and only if $\underline{f} = \underline{g}$.

You don't have to give formal proofs that this is the case, but I suggest that you do.

(2) Consider the following $\mathcal{L}$-formulas:

$$\psi_1 : \forall x \forall y \underline{f}(x) \doteq \underline{g}(y) \quad \psi_2 : \exists x \forall y \underline{f}(x) \doteq \underline{g}(y)$$
$$\psi_3 : \forall x \exists y \underline{f}(x) \doteq \underline{g}(y) \quad \psi_4 : \exists x \exists y \underline{f}(x) \doteq \underline{g}(y)$$

Find $\mathcal{L}$-structures $\mathcal{M}_1, \dots, \mathcal{M}_4$ and $\mathcal{N}_1, \dots, \mathcal{N}_4$ such that $\mathcal{M}_i \vDash \psi_i$ and $\mathcal{N}_i \vDash \neg\psi_i$ for each $i \leq 4$.

Question 3. [20 marks] Let $\mathcal{L}$ be a language with a single unary relation symbol $\underline{P}$ and a single binary relation symbol $\underline{R}$. Consider the following $\mathcal{L}$-sentences:

$$\phi_1 : \exists x \forall y \exists z ((\underline{P}(x) \to \underline{R}(x, y)) \land (\underline{P}(y) \land \neg \underline{R}(y, z)))$$
$$\phi_2 : \exists x \exists z ((\underline{R}(z, x) \to \underline{R}(x, z)) \to \forall y \underline{R}(x, y))$$
$$\phi_3 : \forall y (\exists z \forall t \underline{R}(t, z) \land \forall x (\underline{R}(x, y) \to \neg \underline{R}(x, y)))$$
$$\phi_4 : \exists x \forall y ((\underline{P}(y) \to \underline{R}(y, x)) \land (\forall u)((\underline{P}(u) \to \underline{R}(u, y)) \to \underline{R}(x, y)))$$

For each of these formulas determine whether or not it is satisfied in each of the following $\mathcal{L}$-structures:

(1) A structure with universe $\mathbb{N}$, where $\underline{R}$ is interpreted as the usual order $\leq$ and $\underline{P}$ is the set of all even numbers.

(2) A structure with universe $\mathcal{P}(\mathbb{N})$, where $\underline{R}$ is interpreted as $\subseteq$ and $\underline{P}$ is the set of all finite subsets of $\mathbb{N}$.

(3) A structure with universe $\mathbb{R}$ where $\underline{R}$ is interpreted as the relation $\{(x, y) \in \mathbb{R}^2 : x = y^2\}$ and $\underline{P}$ is interpreted as the set of all rational numbers.

Question 4. [40 marks] In all the languages considered in this exercise $\underline{R}$ is a binary relation symbol, $\underline{\star}, \underline{\oplus}$ are binary function symbols, and $\underline{c}, \underline{d}$ are constant symbols.

(1) In each of the following five cases you are given a language $\mathcal{L}_i$ and two structures $\mathcal{M}_i$ and $\mathcal{N}_i$. For each $i \leq 5$ find an $\mathcal{L}_i$-sentence which is true in $\mathcal{M}_i$ but false in $\mathcal{N}_i$.

(1)	$\mathcal{L}_1 = \{\underline{R}\}$	$\mathcal{M}_1 = (\mathbb{N}; \leq)$	$\mathcal{N}_1 = (\mathbb{Z}; \leq)$
(2)	$\mathcal{L}_2 = \{\underline{R}\}$	$\mathcal{M}_2 = (\mathbb{Q}; \leq)$	$\mathcal{N}_2 = (\mathbb{Z}; \leq)$
(3)	$\mathcal{L}_3 = \{\underline{\star}\}$	$\mathcal{M}_3 = (\mathbb{N}; \times)$	$\mathcal{N}_3 = (\mathcal{P}(\mathbb{N}); \cap)$
(4)	$\mathcal{L}_4 = \{\underline{c}, \underline{\star}\}$	$\mathcal{M}_4 = (\mathbb{N}; 1, \times)$	$\mathcal{N}_4 = (\mathbb{Z}; 1, \times)$
(5)	$\mathcal{L}_5 = \{\underline{c}, \underline{d}, \underline{\oplus}, \underline{\star}\}$	$\mathcal{M}_5 = (\mathbb{R}; 0, 1, +, \times)$	$\mathcal{N}_5 = (\mathbb{Q}; 0, 1, +, \times)$

(2) For each of the sentences below, in the language $\{\underline{c}, \underline{\oplus}, \underline{\star}, \underline{R}\}$ find a structure which satisfies it and a structure which satisfies its negation:

$\phi_1 :\quad \forall x \forall y \exists z (\neg y \doteq \underline{c} \to x \underline{\oplus} (y \underline{\star} z) \doteq \underline{c})$

$\phi_2 :\quad \forall u \forall v \forall w \exists x (\neg x \doteq \underline{c} \to u \oplus (v \oplus x) \oplus (w \oplus x \star (x \star x)) \doteq \underline{c})$

$\phi_3 :\quad \forall x \forall y \forall z (\underline{R}(x, x) \to \underline{R}(x \underline{\star} z, y \underline{\star} z))$

$\phi_4 :\quad \forall x \forall y \forall z (\underline{R}(x, x) \wedge (\underline{R}(x, y) \wedge \underline{R}(y, z) \to \underline{R}(x, z)) \wedge (\underline{R}(x, y) \to \underline{R}(y, x)))$

$\phi_5 :\quad \forall x \forall y (\underline{R}(x, y) \to \neg \underline{R}(y, x))$.