Homework 4

Question 1. [30 marks] Solve the following exercises from the **Week 4** notes to complete the proof of the Substitution Lemma:

(1) Exercise 3.2.6.

(2) Exercise 3.2.8.

Question 2. [20 marks]. Prove Lemma 3.2.9, namely, show that if ϕ is an $\mathcal{L}$-formula, $y \in \mathsf{Var} \setminus \mathsf{Var}(\phi)$ and $x \in \mathsf{Var}$, then $(\phi[y/x])[x/y]$ is equal to ϕ.

Question 3. [40 marks] Let $\mathcal{L}$ be a first-order language, $\mathcal{M} = (M; \dots)$ an $\mathcal{L}$-structure and $N \subseteq M$ such that:

- For all $\underline{c} \in \mathsf{Const}(\mathcal{L})$ we have that $c^{\mathcal{M}} \in N$.
- For all n-ary $\underline{f} \in \mathsf{Fun}(\mathcal{L})$ and all $a_1, \dots, a_n \in N$ we have $f^{\mathcal{M}}(a_1, \dots, a_n) \in N$.

Define an $\mathcal{L}$-structure on N by setting:

- $c^{\mathcal{N}} = c^{\mathcal{M}}$, for all $\underline{c} \in \mathsf{Const}(\mathcal{L})$;
- $f^{\mathcal{N}} = f^{\mathcal{M}} \restriction_{N^n}$, for all n-ary $\underline{f} \in \mathsf{Fun}(\mathcal{L})$;
- $R^{\mathcal{N}} = R^{\mathcal{M}} \cap N^n$ for all n-ary $\underline{R} \in \mathsf{Rel}(\mathcal{L})$.

You are asked to do the following:

(1) Briefly explain why $\mathcal{N} = (N; \dots)$ is an $\mathcal{L}$-structure.

(2) Show that if $A \subseteq M$ is non-empty then there is a unique substructure $\langle A \rangle$ of $\mathcal{M}$ whose universe contains A and such that every substructure of $\mathcal{M}$ whose universe contains A also contains the universe of $\langle A \rangle$. We call $\langle A \rangle$ the **substructure generated by** A.

(3) Show that if $A = \emptyset$ there may not exist such a substructure of $\mathcal{M}$. Give an example, however, in which it exists.

(4) Suppose that $\mathcal{L}$ is **relational** (i.e. it contains no function symbols). What is the universe of $\langle A \rangle$?[1]

(5) We say that a substructure $\mathcal{N}$ of $\mathcal{M}$ is **finitely generated** if it is of the form $\langle A \rangle$ for a finite set $A \subseteq M$. Let ϕ be a sentence of the form $(\forall x_1) \dots (\forall x_n)\psi$, where ψ is a Boolean combination of atomic formulas. Show that $\mathcal{M} \vDash \phi$ if and only if for every finitely generated substructure $\mathcal{N}$ of $\mathcal{M}$ we have that $\mathcal{N} \vDash \phi$.

[1]Observe that $\langle A \rangle$ is *only* defined when A is non-empty.

(6) Show that the result from 5 need not hold when ϕ is of the form $(\exists x)\psi$.

Question 4. [10 marks] Let $\mathcal{L}$ be a language and ϕ an $\mathcal{L}$-sentence. The **finite spectrum** of ϕ is the set $\{n \in \mathbb{N} : \text{ there is } \mathcal{M} = (M; \dots) \vDash \phi, |M| = n\}$, i.e. the set of cardinalities of finite models of ϕ. For each of the following subsets of $\mathbb{N}$ find a formula whose spectrum is exactly that set, or explain why it's not possible.

(1) $\emptyset$.

(2) $\mathbb{N}$.

(3) $\mathbb{N}_{>1}$

(4) $\{n\}$, for $n \in \mathbb{N}_{>1}$