Homework 5

Question 1. [30 marks] Prove that (Q3) is an obsolete axiom. Namely, solve Exercise 3.1.3, from **Week 5** of the notes.

[*Hint.* You are allowed to use instances of propositional tautologies as "propositional axioms". The following tautologies may be useful $(A \to \neg B) \to (B \to \neg A)$ and $(A \to B) \to ((B \leftrightarrow C) \to (A \to C))$.]

Question 2. [40 marks] Write the proof of the Soundness Theorem. This involves the following steps:

(1) Exercise 2.1.1,

(2) Exercise 2.2.1

(3) Proving that the deduction rules are sound, namely, for any sentence ϕ:

- If $T \vDash \phi$ and $T \vDash \phi \to \psi$ then $T \vDash \psi$.
- If $T \vDash \phi$ then $T \vDash (\forall x)\phi$.

[*Hint.* The second item here is not that hard.]

Question 3. [30 marks] Let $\mathcal{L}$ be a language and $\mathsf{F}(\mathcal{L})$ the set of all $\mathcal{L}$-formulas.

(1) Show that there exists at least one map $\mathsf{val} : \mathsf{F}(\mathcal{L}) \to \{T, F\}$ such that:

  (a) If $\phi \in \mathsf{F}(\mathcal{L})$ is of the form $(\forall x)\psi$ then $\mathsf{val}(\phi) = F$.

  (b) If $\phi \in \mathsf{F}(\mathcal{L})$ is of the form $(\exists x)\psi$ then $\mathsf{val}(\phi) = T$.

  (c) If $\phi \in \mathsf{F}(\mathcal{L})$ is of the form $\neg\psi$ then $\mathsf{val}(\phi) = f_\neg(\mathsf{val}(\psi))$.

  (d) If $\phi \in \mathsf{F}(\mathcal{L})$ is of the form $\psi \Box \chi$ then $\mathsf{val}(\phi) = f_\Box(\mathsf{val}(\psi), \mathsf{val}(\chi))$, where $\Box \in \{\wedge, \vee, \to\}$.

Now, let val be as above.

(2) Show that if ϕ is an instance of an axiom (A1)-(A3),(Q1),(Q2),(Q4) (we have already seen (Q3) is obsolete), then $\mathsf{val}(\phi) = T$.

(3) Show that if $\mathsf{val}(\phi \to \chi) = T$ and $\mathsf{val}(\phi) = T$, then $\mathsf{val}(\chi) = T$.

(4) Show that if there exists a derivation of ϕ which does not use the equality axioms or (Gen), then $\mathsf{val}(\phi) = T$. Conclude that there are sentences which do not mention $\doteq$ that are derivable but are not derivable without (Gen).

(5) *Extra:* Can you adapt the argument above to show that the axiom (Q2) is indispensable?[1]

[1]Explaining perhaps typos in a previous version of this HW.