Homework 6

Question 1. [70 marks] Complete the proof of the Completeness Theorem:

(1) Exercise 4.1.2.

(2) Exercise 4.1.7.

(3) Exercise 4.2.2.

(4) Exercise 5.1.4.

The numbering refers to **Week 5** of the notes.[1]

Question 2. [30 marks] Use the compactness theorem to prove the following:

(1) There is no $\mathcal{L}$-theory T (for any language $\mathcal{L}$) such that for all $\mathcal{L}$-structures $\mathcal{M}$ we have that $\mathcal{M} \vDash T$ if and only if $\mathcal{M}$ is finite.

(2) A graph $\mathcal{G} = (V; R^{\mathcal{G}})$ (i.e. an $\mathcal{L}_{\mathsf{graph}}$-structure in which R is irreflexive and symmetric) is called k**-colourable** (for some $k \in \mathbb{N}$) if there is a function $f : V \to k$ such that for all $u, v \in E$ we have that if $R^{\mathcal{G}}(u, v)$ then $f(u) \neq f(v)$. Show that an infinite graph is k-colourable if and only if all of its finite subgraphs are.

[*Hint.* The compactness theorem also holds for uncountable languages. If you feel weird about that, you may assume that $\mathcal{G}$ is countable.]

Question 3. [0 marks] Go over the proof of Theorem 6.2.5 (from **Week 6** of the notes).[2]

[1] Once you see that certain arguments are identical to arguments you've already given before, you can just explain this. The point of this question is not to get you to write a lot of pages of text, but rather to understand the technicalities involved in proving the Completeness theorem.

[2] This used to be a real question, but I realised that Q1 is rather long.