Practice Final Exam 2

This exam consists of 6 questions. The first four questions count for 80% of the exam and cover material you have seen in the lectures. Questions 5 and 6 count only for 20% of the exam and *build* on the material that you have seen in class. I recommend you start with the first four questions.

Question 1 – Basic Definitions

(a) [10 marks] Let $\mathcal{L}$ be a first-order language. Define the sets of (i) $\mathcal{L}$-terms, (ii) atomic $\mathcal{L}$-formulas, (iii) $\mathcal{L}$-formulas. What is an $\mathcal{L}$-sentence? [You do not need to define $\mathsf{Free}(\phi)$ and $\mathsf{Bound}(\phi)$]

(b) [5 marks] Prove that if $\mathcal{M}$ is an $\mathcal{L}$-structure, $\underline{c}$ a constant symbol in $\mathcal{L}$ and $\mathcal{M} \models \phi(\underline{c})$, then $\mathcal{M} \models (\exists x)\phi(x)$. [You need to appeal to Tarski's definition of truth.]

(c) [5 marks] Let T be a first-order $\mathcal{L}$-theory. Define the following terms:

- T is *consistent*.
- T is *complete*.

(d) [20 marks] Give examples of (i) an inconsistent theory, (ii) a consistent theory, (iii) an incomplete theory, (iv) a complete theory. Briefly justify your answers.

Question 2 – Completeness and Compactness

For part (c) of this question, you may assume that $\mathcal{L}$ be a countable language.

(a) [10 marks] Let $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_n \subseteq \cdots$ be a chain of $\mathcal{L}$-theories. Show that if each T_i is consistent, then $T = \bigcup_{n \in \mathbb{N}} T_n$ is consistent.

(b) [5 marks] Define what it means for a first-order theory T to have *Henkin witnesses.*

(c) [10 marks] How are axioms (E1)-(E5) used in the proof that if T is a consistent theory with Henkin witnesses then T has a model?

(e) [15 marks] Prove that there is no first-order theory T such that $\mathcal{M} \models T$ if, and only if, M is finite.

Question 3 – Recursion Theory

(a) [5 marks] Define the *bounded* μ operator.

(b) [10 marks] Prove that if $A, B \subseteq \mathbb{N}$ are primitive recursive, then so are $A \cup B$, $A \cap B$ and $\mathbb{N} \setminus A$.

(c) [5 marks] Define what it means for a set $A \subseteq \mathbb{N}$ to be recursively enumerable.

(d) [20 marks] Let $A \subseteq \mathbb{N}$. Prove that if A and $\mathbb{N} \setminus A$ are both recursively enumerable, then A is recursive. [You may use any results from the course, except of course, the theorem of the complement.]

Question 4 – Incompleteness

(a) [5 marks] Let $\mathcal{N} \vDash T_{PA_0}$. Define the binary relation $\leq$. Suppose that $a, b \in N$ are non-standard elements, does $\mathcal{N} \vDash (a \leq b) \vee (b \leq a)$? [A simple yes or no will suffice.]

(b) [10 marks] Prove that there are models of T_{PA_0} with non-standard elements.

(c) [5 marks] Prove that if $A, B \subseteq \mathbb{N}$ are representable sets, then so are $A \cup B$, $A \cap B$ and $\mathbb{N} \setminus A$. [You may use your answer in 3(b).]

(d) [15 marks] Suppose that $f : \mathbb{N}^2 \to \mathbb{N}$ is representable and the function $g : \mathbb{N} \to \mathbb{N}$ given by

$$g(x) := \mu y.[f(x, y) = 0]$$

is total. Write down a formula showing that g is representable. [You do not need to formally prove that your formula works.]

(e) [5 marks] State Gödel's first incompleteness theorem.

Question 5

(a) [10 marks] Let $\mathcal{L}$ be a first-order language with a single binary relation symbol E, and let T be an $\mathcal{L}$-theory asserting that E is an equivalence relation with infinitely many equivalence classes.

   i. An *axiomatisation of* T is a theory T' such that every model of T is a model of T' and every model of T' is a model of T. Write down an axiomatisation of T.

   ii. Show that T has two countable models which are non-isomorphic.

(b) [10 marks] A first-order theory is called *finitely axiomatisable* if it has a finite axiomatisation (see part (a)i. of this question). Prove that if T is finitely axiomatisable, then there is a finite $T' \subseteq T$ which is an axiomatisation of T.

Question 6

(a) [10 marks] Let $\mathcal{M} \models T_{PA}$ and assume that $\mathcal{M}$ has the standard model as a proper subset. Define a relation $\sim$ on M^2 by setting $x \sim y$ if and only if there exist $m, n \in \mathbb{N}$ such that:

$$\mathcal{M} \models x \,\underline{+}\, \underline{n} \;\doteq\; y \,\underline{+}\, \underline{m}.$$

i. Prove that $\sim$ is an equivalence relation.

ii. Prove that if $a \sim a'$ and $b \sim b'$ then $a +^{\mathcal{M}} b \sim a' +^{\mathcal{M}} b'$.

(b) [10 marks] Recall that Drv is the set:

$$\{(\#\phi, \#\#\Delta) \in \mathbb{N}^2 : \Delta = (\delta_1, \ldots, \delta_n) \text{ is a derivation of } \phi \text{in } T_{PA_0}\}$$

We have shown in lectures that this set is representable. Let $\delta(x, y)$ be an $\mathcal{L}_{Peano}$-formula representing it.

Are the following assertions true or false, for an $\mathcal{L}_{Peano}$-sentence ϕ:

i. $\mathbb{N} \models (\exists x)\delta(\#\phi, x) \to \phi$.

ii. $\mathbb{N} \models \phi \to (\exists x)\delta(\#\phi, x)$.

Carefully justify your answers.