CHAPTER 1

We all have to start somewhere

1. The set-up

[INT. Second International Congress of Mathematicians, Paris]

The year is 1900, and HILBERT is about to deliver his famous address *"Mathematical Problems"*. He walks up to the stage. The air in the room is electric.

HILBERT

> Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries? What particular goals will there be toward which the leading mathematical spirits of coming generations will strive? What new methods and new facts in the wide and rich field of mathematical thought will the new centuries disclose?
>
> History teaches the continuity of the development of science. We know that every age has its own problems, which the following age either solves or casts aside as profitless and replaces by new ones [...]

FADE OUT. The talk continues momentarily in VOICE OVER but the words slowly get less and less discernible.

Picture yourself sitting in the audience, worried that the whole speech will continue in this extremely verbose manner. Will we last even an hour of this? The schedule of the ICM said that he'd discuss problems, why is he blabbering on...

FADE IN.

HILBERT [CT'D]

[...] mathematical ideas
come up, the problem arises
for mathematical science to
investigate the principles
underlying these ideas and so to
establish them upon a simple and
complete system of axioms, that
the exactness of the new ideas and
their applicability to deduction
shall be in no respect inferior
to those of the old arithmetical
concepts[...]

Ah, now we're getting somewhere! Honestly, David is a lovely fella, but I don't think I have the patience to sit through this. I'll just wait for the write-up...

A year later, the write-up is eventually published, and in it there are twenty-three (24) listed problems. You start reading:

(1) Cantor's problem of the cardinal number of the continuum.

(2) The compatibility of the arithmetical axioms.

...

There are more problems on the list, but let's take things one item at a time. These two problems are somehow different from the remaining twenty-one (22), especially the second problem is really a fundamental problem. It's not so much a problem of mathematics, but more so a problem about "how to do mathematics". This is where mathematical logic comes in.

2. So what about logic?

To start talking about logic from the beginning, we'd have to go back to the bronze age and the origins of written text (I think, don't fact check me). But to be fair, we don't have to go all that back to see how "logic" can show up in our day-to-day life – it should be enough for the purposes of this section to go back to the distant

Year of Our COVID, 2020. You turn on your TV (in the year 2020) and you hear an argument:

"All cops are pigs.", says a panellist and "Derek Chauvin is a cop." The other panellist nods and agrees: "Derek Chauvin is a pig."

Of course, if your TV turns on on the wrong channel, you may end up hearing a panellist convey that conversation saying: "Babe is a pig..." and (implicitly deducing "Babe is a cop") following this up with "...and they want to defund Babe". Won't somebody please think of the children!

The first deduction feels much more valid than the second one, and not just because you heard the second one on Fox – poor Babe never hurt anyone or participated in systemic violence. That's essentially what we're here to discuss: How can we abstract away all the minor details that make arguments hard to parse and understand if they are valid or not? But well, don't get your hopes too high, this will still be a mathematics class, and we will talk not about logic in general, what we'll focus on will be *mathematical* logic.

Mathematical logic is, well... more mathematical. The arguments that mathematical logic wants to abstract are the usual arguments you've already seen in other mathematics courses. What if there were a framework in which we could express all those ε-δ proofs? And also all those proofs about matrices. And also all those proofs from that other course that every freshman has to tackle. Mathematical logic tries to be the language of mathematics. But in being so, it is also a branch of mathematics, and hence it's a language that can (and must) speak of itself. We've reached our first point of self-reference:

All Cretan's are liars, said Epimenides, the Cretan.

Our exploration of mathematical logic will mainly focus on mathematical logic (who'd have thunk), by which I mean both the tautological contentless sentence that you probably though I meant, but also that we will develop several logical tools, which could be applied to various branches of mathematics, but in these notes, but, we will apply them mainly to mathematical logic itself!

3. About these notes

Spread out in these notes are many exercises and, at least in my opinion, solving them, or at least looking at them long and hard enough until you know that you know how to solve them, is essential to understanding what's going on. Sometimes we all have to get our hands dirty, don't we? Each of the main chapters of the course also comes with a number of HW sheets, which you will need to submit (these are

made up of a selection of exercises from the chapter they are in as well as some long-ish, interesting problems).

There are also several subsections (or subsubsections) in these notes whose title is marked in blue. These will not be covered in detail in class and contain either (a) material that is beyond the scope of the course; (b) annoyingly technical details; or (c) long digressions that *I* think are interesting, but your mileage may vary (also, covering everything I think is interesting would definitely be a recipe for disaster).

Finally, let me also mention here that I am a biased logician – my field of study is the branch of mathematical logic called *model theory*.[1] When writing these notes, I tried to suppress my biases as much as possible, and I think I succeeded (but I may have overdone it a bit). In any case, there is hardly any mention of the phrase "model theory" outside of this paragraph.

4. Back to our story

[INT. International Congress of Mathematics]

Hilbert has finished his talk and walks off the stage to a standing ovation. CUT TO BLACK.

[EXT. Café in Vienna. DAY]

Hilbert is talking to a young GÖDEL.

HILBERT

I'll be damned! I think I gave you guys enough work for at least a century. But honestly, Kurt, the only real problem from my list is Problem 2. I swear to god, if in whatever way some mad genius shows this problem can't be solved, I'll quit mathematics and open a hotel.

We cut to Gödel who appears to be deep in thought.

[1] If you are interested in this, you could consider MATH712, or asking me about it.