CHAPTER 6

Okay it's undecidable, but it can't be incomplete too (Cont'd)

2. Some dirty work

2.1. Representation... Recall that $\mathcal{F}_p$ is the set of all (total) functions from $\mathbb{N}^p$ to $\mathbb{N}$. We will start by defining a weak form of definability, for such functions:

Definition 2.1.1. Let $f \in \mathcal{F}_p$ and $\phi(x, y_1, \ldots, y_p)$ be an $\mathcal{L}_{Peano}$ formula. We say that ϕ *represents* f if for all $n_1, \ldots, n_p \in \mathbb{N}$ we have that:

$$T_{PA_0} \vdash (\forall x)(\phi(x, \underline{n}_1, \ldots, \underline{n}_p) \leftrightarrow x \doteq \underline{f(n_1, \ldots, n_p)}).$$

In this case, we say that f is *representable.*

The point is that ϕ represents f if and only if for every model $\mathcal{N}$ of weak Peano arithmetic and every p-tuple $\bar{n}$ (of actual factual) naturals we plug into f (well f can only take actual factual naturals as inputs), there is a unique element in N satisfying $\phi(x, \bar{n})$ and that element is STANDARD, but not only that, it also is the correct element, in that it interprets the term $\underline{f(n_1, \ldots, n_p)}$).

Of course, a subset $A \subseteq \mathbb{N}^p$ is called **representable** if its characteristic function is representable. Equivalently, $A \subseteq \mathbb{N}^p$ is representable if and only if there is an $\mathcal{L}_{Peano}$-formula $\phi(x_1, \ldots, x_p)$ such that for all $\bar{n} \in \mathbb{N}^p$ we have:

- If $\bar{n} \in A$ then $T_{PA_0} \vdash \phi(\bar{n})$.
- If $\bar{n} \notin A$ then $T_{PA_0} \vdash \neg\phi(\bar{n})$.

It should be clear why these two notions are equivalent, but just to be safe:

Let $A \subseteq \mathbb{N}^p$.

- If ϕ represents A, then the formula:
$$(\phi(x_1, \ldots, x_p) \wedge x \doteq \underline{1}) \vee (\neg\phi(y_1, \ldots, y_p) \wedge x \doteq \underline{0})$$
represents $\mathbb{1}_A$.
- If $\mathbb{1}_A$ is representable by the formula $\psi(x, y_1, \ldots, y_p)$, then, the formula $\psi(1, y_1, \ldots, y_p)$ represents A.

You may be wondering, at this point, why didn't I write:

$$\bar{n} \in A \text{ if and only if } T_{PA_0} \vdash \phi(\bar{n}).$$

Is it because I'm trying my hardest to up the page count of these notes? Well, no! The point is that we don't know if T_{PA_0} is complete or not at this point (but if you're reading between the lines we actually know that it's not!) so we can't quite say that:

$$T_{PA_0} \nvdash \phi(\bar{n}) \text{ if and only if } T_{PA_0} \vdash \neg\phi(\bar{n}).$$

We are asking, thus, for something strong. Not only that T_{PA_0} proves membership in A (when it has to) but also that it *proves* non-membership in A (when it has to).

Here are some examples of representable functions:

(1) The successor function.

(2) The constant function 0.

(3) The projection functions.

We essentially proved all of these in the proof that the map:

$$f : \mathbb{N} \to M$$
$$n \mapsto \underline{n}^{\mathcal{N}}$$

is an injective homomorphism (where M is the set of standard elements of some model $\mathcal{N} \models T_{PA_0}$).

Do you see where I'm going with this?

Theorem 2.1.2 (The Representation Theorem). *Every total recursive function is representable.*

Going back to our proof that all register machine computable functions are recursive, we actually wrote down an explicit formula:

$$g(x_1, \ldots, x_p) = \mathsf{return}(\mathsf{Run}_R\text{-}\mathsf{at}(\mathsf{term\text{-}time}_R(x_1, \ldots, x_p), x_1, \ldots, x_p)).$$

and this formula is clearly built using the μ operator, constant functions, projections and primitive recursive functions. Of course, as we already know, all recursive functions are (register machine) computable, and thus, given a recursive function, we can find the register machine that computes it and then reconstruct the function we started with using the formula above.

We are now visibly interested in *total* functions. Let's define closure under the *total* unbounded μ operator. Let $\mathcal{R}_{\mathsf{total}}$ be the smallest subset of $\mathcal{F}$ which contains the basic functions, is closed under primitive recursion, composition, and moreover is closed under the following (which we dub the total unbounded μ operator):

> Let $A \subseteq \mathbb{N}^{n+1}$ be such that $\mathbb{1}_A \in \mathcal{R}$ and such that the function:
> $$f(\bar{x}) := \mu y.[(y, \bar{x}) \in A],$$
> is total. Then $f \in \mathcal{R}_{\mathsf{total}}$.

It is not hard to see that $\mathcal{R}_{\mathsf{total}} \subseteq \mathcal{R}$ and, in fact, $\mathcal{R}_{\mathsf{total}} = \mathcal{R} \cap \mathcal{F}$.

What's the point of all of this? Well, in showing that every total recursive function is representable, it will suffice to show that:

(1) The set of representable functions contains all the basic functions (Ha! We did this already.)
(2) The set of representable functions is closed under composition.
(3) The set of representable functions is closed under the total μ operator.
(4) The set of representable functions is closed under primitive recursion.

We'll discuss points (2) and (3) here and point (4) in the next section, which will be rather technical and subtle.

Proposition 2.1.3. *The set of representable functions is closed under composition.*

PROOF. Let $f_1, \ldots, f_n \in \mathcal{F}_p$ and $g \in \mathcal{F}_n$ be representable functions, and suppose that for $1 \leq i \leq n$, the formula $\phi_i(x, y_1, \ldots, y_p)$ represents f_i and that the formula $\psi(x, y_1, \ldots, y_n)$ represents g. Then, $g(f_1, \ldots, f_n)$ is represented by:

$$(\exists z_1) \cdots (\exists z_n) \left(\psi(y, z_1, \ldots, z_n) \wedge \bigwedge_{i=1}^{n} \phi_i(z_i, x_1, \ldots, x_p) \right).$$

□

The μ business is harder, but we can do it!

Proposition 2.1.4. *The set of representable functions in closed under the total μ operator.*

PROOF. We have to show that if $A \subseteq \mathbb{N}^{p+1}$ is representable and the function $f(x_1, \ldots, x_p)$ given by $\mu y.[(y, x_1, \ldots, x_p) \in A]$ is total, then f is representable.

Suppose that $\phi(y, x_1, \ldots, x_p)$ represents A. I claim that the formula $\psi(x, y_1, \ldots, y_p)$ given by

$$\phi(y, x_1, \ldots, x_p) \wedge ((\forall z)(z < y \rightarrow \neg\phi(z, x_1, \ldots, x_p)))$$

represents f. What does this mean? Well... It means that given any model $\mathcal{M} \models T_{PA_0}$, the interpretation b of $\underline{f(n_1, \ldots, n_p)}$ in $\mathcal{M}$ is the only element of $\mathcal{M}$ that satisfies the formula $\psi(x, \underline{n_1}, \ldots, \underline{n_p})$. Obviously, since $\mathcal{M} \models T_{PA_0}$ and ϕ represents f we have that

$$\mathcal{M} \models \phi(b, \underline{n}_1, \ldots, \underline{n}_p).$$

Also, if $c \in M$ is smaller than b, then (since b was standard) c is standard, so by definition of f we have that:

$$\mathcal{M} \models \neg\phi(c, \underline{n}_1, \ldots, \underline{n}_p).$$

In particular, we have that:

$$\mathcal{M} \models \psi(b, \underline{n}_1, \ldots, \underline{n}_p).$$

Finally, if $d \in M$ also satisfies $\psi(x, \underline{n}_1, \ldots, \underline{n}_p)$, then neither $d < b$ nor $b < d$ can hold in $\mathcal{M}$, but since b is standard, we must have that $d = b$. □

Now, for the crux of the argument, we have to prove the following:

Proposition 2.1.5. *Let $f \in \mathcal{F}_p$ and $g \in \mathcal{F}_{p+2}$ be representable functions. Then, the function h defined by recursion from f and g, by:*

$$h(x_1, \ldots, x_k, x_{k+1}) = \begin{cases} f(x_1, \ldots, x_k) & \textit{if } x_{k+1} = 0 \\ g(x_1, \ldots, x_k, x_{k+1} - 1, h(x_1, \ldots, x_k, x_{k+1} - 1)) & \textit{otherwise} \end{cases}$$

is also representable. In particular, the set of representable functions contains all primitive recursive functions.

We will skip the proof of this proposition, for now, because it's rather long and technical. We may come back to it once we've finished the rest of the proof of the incompleteness theorem.