CHAPTER 6

Okay it's undecidable, but it can't be incomplete too (Cont'd)

4. I'm incompletely baffled

Starting off with a bang:[1]

THEOREM 4.0.1. *Let T be a consistent theory extending T_{PA_0}. Then, T is undecidable.*

PROOF. Suppose not,[2] and play the following diagonal game with me:

Define the set:

$$\Theta = \{(m, n) : m \in \mathsf{Fml}(x_1) \text{ and } T \vdash \phi(\underline{n})\},$$

where $\mathsf{Fml}(x_1)$ is the set of all $\mathcal{L}$-formulas with at most one free variable, that variable being x_1. Since T is recursive, the set Θ is recursive. Indeed, we have that $(m, n) \in \Theta$ if and only if

$$m = \#\psi(x_1) \in \mathsf{Fml}(x_1) \text{ and } \#[\psi(\underline{n})] \in \{\#\phi : \phi \in \mathsf{Sen}(\mathcal{L}) \text{ and } T \vdash \phi\}$$

Intuitively, Θ contains the collection of things that T can prove about individual standard terms. Before we get actually diagonal, let's also observe that the function $\lambda x.\#\underline{x}$ is a primitive recursive function, since it is the function f defined as follows

$$f(0) = \mathsf{tuple}^3(0, 0, 0)$$

and:

$$f(x + 1) = \#(\underline{S}\ f(x)) = \mathsf{tuple}^3(f(x), 0, 1),$$

Let's remark also that:

$$(m, n) \in \Theta \iff m \in \mathsf{Fml}(x_1) \text{ and } \mathsf{subst}_{fla}(1, \#\underline{n}, m) \in \{\#\phi : \phi \in \mathsf{Sen}(\mathcal{L}) \text{ and } T \vdash \phi\}.$$

Thus, the set:

$$\mathsf{D} = \{n \in \mathbb{N} : (n, n) \notin \Theta\}$$

[1]Before the bang, I would suggest going back and reminding yourselves about representable functions.

[2]That is, suppose that $\{\#\phi : \phi \in \mathsf{Sen}(\mathcal{L}) \text{ and } T \vdash \phi\}$ is recursive

is also recursive (just trace it back in the previous equivalence). But now, by the Representation Theorem, this set is representable. Explicitly, there is an $\mathcal{L}_{Peano}$-formula $\psi(x_0)$ such that for all $n \in \mathbb{N}$ we have that:

- If $n \in \mathsf{D}$ then $T_{PA_0} \vdash \psi(\underline{n})$, and hence $T \vdash \psi(\underline{n})$.
- If $n \notin \mathsf{D}$ then $T_{PA_0} \vdash \neg\psi(\underline{n})$, and hence $T \vdash \neg\psi(\underline{n})$.

Now, we can, and we must, consider the Gödel number of the formula $\psi(x_1)$! Of course, $\#\psi(x_1) \in \mathsf{Fml}(x_1)$. If $\#\psi(x_1) \in \mathsf{D}$ then on the one hand we have that $(\#\psi(x_1), \#\psi(x_1)) \notin \Theta$ so

$$T \nvdash \psi(\#\psi(x_1)),$$

but on the other contradictory hand, since $\#\psi(x_1) \in \mathsf{D}$ then:

$$T \vdash \psi(\#\psi(x_1)),$$

which is a contradiction. Hence $\#\psi \notin \mathsf{D}$. But then, on the one hand we have that $(\#\psi, \#\psi) \in \Theta$, so

$$T \vdash \psi(\#\psi(x_1)).$$

Again, on the other hand, since $\#\psi(x_1) \notin \mathsf{D}$

$$T \vdash \neg\psi(\#\psi(x_1)),$$

and we have reached a contradiction again, because we assumed that T was consistent! $\square$

THEOREM 4.0.2 (First Incompleteness Theorem). *Let T be a consistent recursive theory that includes T_{PA_0}. Then T is incomplete.*

PROOF. If T were complete, then it would be decidable! $\square$

Okay great, but this theorem by itself just tells us that there exists some $\mathcal{L}_{Peano}$-formula which is neither derivable nor refutable from Peano's axioms. If we were to follow all the steps of the proof super carefully, we could in fact write down that formula. But of course this would not tell us if this formula has a meaning and if it does, what that meaning is. The second incompleteness theorem is the real heart of the cheese. It provides us with a concrete and meaningful formula that T_{PA} cannot prove. That formula?

Oh but the formula that expresses that T_{PA} is consistent! We are so very close to the proof of the second incompleteness theorem now, but unfortunately it needs more work than we have time to do in this class.

We'll close these notes with Church's theorem:

THEOREM 4.0.3. *The set:*

$$T_0 = \{\phi : \phi \textit{ is a universally valid } \mathcal{L}\textit{-formula}\}$$

is not recursive.

PROOF. Let $\psi = \bigwedge T_{PA_0}$. Then for every $\mathcal{L}_{Peano}$-formula ϕ we have that:

$$T_{PA_0} \vdash \phi \iff (\psi \to \phi) \in T_0.$$

So it T_0 were recursive, T_{PA_0} would be decidable. □