CHAPTER 4

Sounds like things are complete (Cont'd)

6. Back to Semantics: Baby's first steps in model theory

We've learned a lot about $\mathcal{L}$-structures (for a fixed first-order language $\mathcal{L}$), but since we will at some point try to do mathematics with them, besides just $\mathcal{L}$-structures it's important to see how $\mathcal{L}$-structures are related, that is, we really need to discuss *"$\mathcal{L}$-maps"* (whatever these may be).

In HW4 Q.3 we defined the notion of an $\mathcal{L}$-substructure. Let's talk a bit more about maps now.

6.1. Embeddings of the basic and the elementary kind. If $\mathcal{L}$-structures generalise groups and graphs and fields and vector spaces, or whatever, then $\mathcal{L}$-maps really should generalise group and graph and field and vector space or whatever "morphisms" (so group homomorphisms, graph homomorphisms, field embeddings, linear maps or whatever).

Definition 6.1.1. Let $\mathcal{L}$ be a first-order language and $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. A map $h : M \to N$ is an $\mathcal{L}$-*homomorphism* (sometimes denoted $h : \mathcal{M} \to \mathcal{N}$) if:

(1) For all $\underline{c} \in \mathsf{Const}(\mathcal{L})$ we have that $h(c^{\mathcal{M}}) = c^{\mathcal{N}}$.

(2) For all n-ary $\underline{R} \in \mathsf{Rel}(\mathcal{L})$ we have that:
$$\text{If } R^{\mathcal{M}}(a_1, \ldots, a_n) \text{ then } R^{\mathcal{N}}(h(a_1), \ldots, h(a_n)),$$
for all $a_1, \ldots, a_n \in M$.

(3) For all n-ary $\underline{f} \in \mathsf{Fun}(\mathcal{L})$ we have that:
$$h(f^{\mathcal{M}}(a_1, \ldots, a_n)) = f^{\mathcal{N}}(h(a_1), \ldots, h(a_n)),$$
for all $a_1, \ldots, a_n \in M$.

We call a homomorphism $h : \mathcal{M} \to \mathcal{N}$ an $\mathcal{L}$-*embedding* if h is an injective $\mathcal{L}$-homomorphism which satisfies the following stronger version of (2):

(2)' For all n-ary $\underline{R} \in \mathsf{Rel}(\mathcal{L})$ we have that:

$$R^{\mathcal{M}}(a_1, \dots, a_n) \text{ if and only if } R^{\mathcal{N}}(h(a_1), \dots, h(a_n)),$$

for all $a_1, \dots, a_n \in M$.

Exercise 6.1.2. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures, with $M \subseteq N$. Show that $\mathcal{M}$ is a substructure of $\mathcal{N}$ if an only if the inclusion map $\iota : M \to N$ is an $\mathcal{L}$-embedding.

Definition 6.1.3. An $\mathcal{L}$-*isomorphism* is just a surjective $\mathcal{L}$-embedding. If there is an $\mathcal{L}$-isomorphism from an $\mathcal{L}$-structure $\mathcal{M}$ to an $\mathcal{L}$-structure $\mathcal{N}$, then we say that $\mathcal{M}$ and $\mathcal{L}$ are $\mathcal{L}$-*isomorphic*, denoted by $\mathcal{M} \simeq \mathcal{N}$. An $\mathcal{L}$-*automorphism* is an $\mathcal{L}$-isomorphism from an $\mathcal{L}$-structure $\mathcal{M}$ to itself.

Exercise 6.1.4. Let $\mathsf{Aut}(\mathcal{M})$ denote the set of all $\mathcal{L}$-automorphisms of an $\mathcal{L}$-structure $\mathcal{M}$. Prove that $\mathsf{Aut}(\mathcal{M})$ is a group, under function composition.

In model theory, we are interested in *definable sets*, that is, subsets of (some Cartesian power of) the domain of an $\mathcal{L}$-structure $\mathcal{M}$ which are given by realisations of some formula, that is, subsets of the form:

$$\{(a_1, \dots, a_n) \in M^n : \mathcal{M} \vDash \phi[a_1, \dots, a_n]\},$$

for some $\mathcal{L}$-formula $\phi(x_1, \dots, x_n)$.

Example 6.1.5. Let $\mathcal{G} = (V; E)$ be a graph, then the set of pairs of vertices which are connected by a path of length 2 is the following definable set:

$$\{(v_1, v_2) \in V^2 : (\exists y)(\underline{E}(x_1, y) \wedge \underline{E}(y, x_2))\}.$$

Exercise 6.1.6. Let $\mathcal{N}$ be the standard model of Peano arithmetic (in the usual language). Show that the set of all primes is definable.

It should be clear that $\mathcal{L}$-embeddings do not necessarily preserve definable sets (in the definition of an $\mathcal{L}$-embedding, we only ask that it preserves sets defined by atomic formulas!). We thus need a stronger version of an embedding.

Definition 6.1.7. Let $\mathcal{L}$ be a first-order language, $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. An $\mathcal{L}$-embedding $h : \mathcal{M} \to \mathcal{N}$ is called *elementary* (or in full an $\mathcal{L}$-*elementary embedding*) if for every $\mathcal{L}$-formula $\phi(x_1, \dots, x_n)$ we have that:

$$\mathcal{M} \vDash \phi[a_1, \dots, a_n] \text{ if and only if } \mathcal{N} \vDash \phi[h(a_1), \dots, h(a_n)].$$

If $M \subseteq N$ and the inclusion map is an $\mathcal{L}$-elementary embedding, then we say that $\mathcal{M}$ is an $\mathcal{L}$-*elementary substructure of* $\mathcal{N}$, which we denote by $\mathcal{M} \preccurlyeq \mathcal{N}$, You have essentially dealt with the following example in HW3:

Example 6.1.8. In the language of groups, $(\mathbb{Z}; 0, +)$ is a substructure of $(\mathbb{Q}, 0, +)$ that is not elementary. Similarly, in the language of fields, $(\mathbb{Q}; 0, 1, +, \times)$ is a substructure of $(\mathbb{R}; 0, 1, +, \times)$ that is not elementary. On the other hand, in the language of linear orders, $(\mathbb{Q}, <)$ is an elementary substructure of $(\mathbb{R}; <)$.

Exercise 6.1.9. Show that every $\mathcal{L}$-isomorphism is an $\mathcal{L}$-elementary embedding.

We now come to a really confusing bit of terminology:

Definition 6.1.10. Let $\mathcal{M}$ and $\mathcal{N}$ be two $\mathcal{L}$-structures. We say that $\mathcal{M}$ is *elementarily equivalent* to $\mathcal{N}$ if for every $\mathcal{L}$-sentence ϕ we have that:

$$\mathcal{M} \vDash \phi \text{ if and only if } \mathcal{N} \vDash \phi.$$

We denote this by $\mathcal{M} \equiv \mathcal{N}$

Remark 6.1.11. Here is a bunch of facts:

(1) If $\mathcal{M} \simeq \mathcal{L}$ then $\mathcal{M} \equiv \mathcal{N}$.

(2) If $\mathcal{M} \preccurlyeq \mathcal{N}$ then $\mathcal{M} \equiv \mathcal{N}$,

(3) It is **not** the case that if $\mathcal{M} \equiv \mathcal{N}$ and $\mathcal{M} \subseteq \mathcal{N}$ (i.e. $\mathcal{M}$ is an $\mathcal{L}$-substructure of $\mathcal{N}$) then $\mathcal{M} \preccurlyeq \mathcal{N}$.

Exercise 6.1.12. Find an example of a substructure $\mathcal{N}$ and a substructure $\mathcal{N}$ which are elementarily equivalent, but so that $\mathcal{M}$ is not an elementary substructure of $\mathcal{N}$.

This is thus a terribly unfortunate choice of words. The problem is that the word *elementary* can refer to both *"basic"* and *"of elements"*. The way to think about it(or at least, the way *I* think about it) is the following:

- Elementary equivalence refers to the "basic" meaning of the word elementary – elementary equivalence is the most basic form of equivalence we can ask.
- An elementary embedding is an embedding that respects what happens to the "elements".

6.2. The Löwenheim-Skolem Theorems. It is clear that the terms elementarily equivalent and elementary substructure do not mean exactly what we'd like them to mean. Our first result (not covered in class, so not examinable) is a test that allows us (rather practically) to determine if a substucture is, in fact, elementary.

THEOREM 6.2.1 (The Tarski-Vaught test). *Let $\mathcal{N}$ be an $\mathcal{L}$-structure and $\mathcal{M}$ a substructre of $\mathcal{N}$. Then, the following are equivalent:*

(1) $\mathcal{M} \preccurlyeq \mathcal{N}$.

(2) For every $\mathcal{L}$-formula $\phi(x_1, \ldots, x_n, y)$ and all elements $a_1, \ldots, a_n \in M$ we have:

If $\mathcal{N} \vDash (\exists y)(\phi[a_1/x_1, \ldots, a_n/x_n])(y)$ *then* $\mathcal{M} \vDash (\exists y)(\phi[a_1/x_1, \ldots, a_n/x_n])(y)$

Remark 6.2.2. The notation $(\exists y)(\phi[a_1/x_1, \ldots, a_n/x_n])(y)$ will get tedious. Let's just write $(\exists y)\phi(a_1, \ldots, a_n, y)$, and pretend we're okay with it.

PROOF. (1) $\implies$ (2) is trivial, so we just do (2) $\implies$ (1). By definition, we have to show that for any formula $\psi(x_1, \ldots, x_m)$ and all $b_1, \ldots, b_m$ in M we have that:

$$\mathcal{M} \vDash \phi(b_1, \ldots, b_m) \text{ if and only if } \mathcal{N} \vDash \phi(b_1, \ldots, b_m).$$

The proof will be by induction on the complexity of ψ and without loss of generality, we may assume that ψ does not contain any $\forall$ quantifiers [WHY?]. The Boolean cases are easy, so we need only worry about the case where ψ starts with an existential quantifier. This case follows easily by inductive hypothesis (i.e. exercise). □

And now for an application of the Tarski-Vaught test:

THEOREM 6.2.3 (Downward Löwenheim-Skolem Theorem). *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$ and suppose that $|\mathcal{M}| \geq |\mathcal{L}| + \aleph_0$.*[1] *Then, there is an elementary substructure $\mathcal{M}_0$ of $\mathcal{M}$ that includes A and such that $|\mathcal{M}_0| = \max\{|A|, |\mathcal{L}| + \aleph_0\}$.*

PROOF. We may assume that $|A| \geq |\mathcal{L}| + \aleph_0$, by arbitrarily adding more elements to A (recall $\mathcal{M}$ is assumed to be big enough for us to do this). The following is just a little bit of counting:

CLAIM 1. If $B \subseteq M$ has $|B| \geq |\mathcal{L}| + \aleph_0$, then $|\langle B \rangle| = |B|$.

PROOF OF CLAIM 1. Exercise. ◀

[1] If you're going about this assuming that $\mathcal{L}$ is countable you need only assume that $\mathcal{M}$ is infinite.

We (inductively) build a chain:

$$A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$$

of subsets of M all of which have cardinality equal to $|A|$. Once A_n has been built, we build A_{n+1} as its closure under existential formulas with parameters. Loads of words, let's be explicit:

- For every formula $\phi(x_1, \ldots, x_m, y)$ and every sequence $a_1, \ldots, a_m \in A_n$, if $\mathcal{M} \vDash \exists y \phi(a_1, \ldots, a_m, y)$, then we choose an element $c_{\phi, a_1, \ldots, a_m} \in M$ witnessing this existential formula, i.e. such that:

$$\mathcal{M} \vDash \phi[a_1, \ldots, a_m, c_{\phi, a_1, \ldots, a_m}].$$

- Let B_n be the following set:

$$B_n = A_n \cup \bigcup \{c_{\phi, a_1, \ldots, a_m}\},$$

  where the big union ranges over all $m \in \mathbb{N}$ and all formulas $\phi(x_1, \ldots, x_m, y)$ and all $a_1, \ldots, a_m \in A_n$ such that $\mathcal{M} \vDash \exists y \phi(a_1, \ldots, a_m, y)$. Observe that $|B_n| = |A_n| = |A|$, by induction.

- Let $A_{n+1} = \langle B_n \rangle$. Then $|A_{n+1}| = |B_n| = |A|$, by the claim and the previous bullet.

Now, take $\mathcal{M}_0 = \bigcup_{n \in \mathbb{N}} A_n$. This clearly has cardinality equal to that of A, so we just need to show that it is an elementary substructure. In fact, it suffices to show that it satisfies (2) in the Tarski-Vaught test. But this is easy by construction, since for any $\mathcal{L}$-formula $\phi(x_1, \ldots, x_m, y)$ and any $a_1, \ldots, a_m \in M_0$ such that:

$$\mathcal{M} \vDash (\exists y) \phi(a_1, \ldots, a_m, y),$$

there is some $n \in \mathbb{N}$ such that $a_1, \ldots, a_m \in A_n$, and thus there is some witness in A_{n+1}. □

Hoorah! We can go *down*. Let's see how we can go *up* (again, the only part of this section that we discussed in class was the main theorem).

Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. We write $\mathcal{L}(A)$ for the language $\mathcal{L}$ expanded by a fresh constant symbol for each element of A, that is $\mathsf{Rel}(\mathcal{L}(A)) = \mathsf{Rel}(\mathcal{L})$, $\mathsf{Fun}(\mathcal{L}(A)) = \mathsf{Fun}(\mathcal{L})$ and $\mathsf{Const}(\mathcal{L}(A)) = \mathsf{Const}(\mathcal{L}) \cup \{\underline{a} : a \in A\}$. We may canonically view $\mathcal{M}$ as an $\mathcal{L}(A)$-structure $\mathcal{M}'$, by interpreting each new constant symbol $\underline{a} \in \mathsf{Const}(\mathcal{L}(A)) \setminus \mathsf{Const}(\mathcal{L})$ as the element it should represent (What?), i.e. $a^{\mathcal{M}'} = a$ (this is where the underlying and superscripting gets confusing).

Then, **elementary diagram** of $\mathcal{M}$, denoted $\mathsf{ElDiag}(\mathcal{M})$ is the set of all $\mathcal{L}(M)$-sentences $\phi(\underline{a}_1, \ldots, \underline{a}_n)$, where $\phi(x_1, \ldots, x_n)$ is an $\mathcal{L}$-formula, $\underline{a}_1, \ldots, \underline{a}_n \in \mathsf{Const}(\mathcal{L}(A))$ and $\mathcal{M} \vDash \phi[a_1, \ldots, a_n]$.[2] We denote this by $\mathsf{ElDiag}(\mathcal{M})$. In future terms, $\mathsf{ElDiag}(\mathcal{M})$ is the "complete theory" of the expansion of $\mathcal{M}$ to the canonical $\mathcal{L}(M)$-structure.

Remark 6.2.4. In the notation above, let $\mathcal{N}'$ be an $\mathcal{L}(M)$-structure such that $\mathcal{N} \vDash \mathsf{ElDiag}(\mathcal{M})$, and let $\mathcal{N}$ be the **reduct** of $\mathcal{N}$ down to $\mathcal{L}$ (i.e. the structure obtained by forgetting all symbols in $\mathcal{L}(M) \setminus \mathcal{L}$). For each $a \in M$, let $g(a)$ denote $a^{\mathcal{N}}$ (i.e. the interpretation of $\underline{a}$ in $\mathcal{N}^\star$). Then, g is an injective [Why?] map from M to N such that for every formula $\phi(x_1, \ldots, x_n)$ of $\mathcal{L}$ we have that:

$$\mathcal{M} \vDash \phi[a_1, \ldots, a_n] \text{ if and only if } \mathcal{N} \vDash \phi[g(a_1), \ldots, g(a_n)],$$

for any $a_1, \ldots, a_n \in M$. Thus, g is an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.[3]

So what's so cool about elementary diagrams?

THEOREM 6.2.5. *Every infinite $\mathcal{L}$-structure $\mathcal{M}$ has a proper elementary extension.*

PROOF. Let $\underline{c}$ be a fresh constant symbol not in $\mathcal{L}(M)$, and consider the theory:

$$\mathsf{ElDiag}(\mathcal{M}) \cup \{\neg(\underline{c} = \underline{a}) : a \in M\}.$$

It suffices to show that this set is satisfiable. This is immediate, by compactness. □

The Upward Löwenheim-Skolem Theorem is the slightly more general version of the theorem above:

THEOREM 6.2.6 (Upward Löwenheim Skolem). *Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure and $\kappa \geq \max\{|\mathcal{M}|, |\mathcal{L}| + \aleph_0\}$. Then there is an elementary extension $\mathcal{N} \succcurlyeq \mathcal{M}$ of $\mathcal{M}$ with $|\mathcal{N}| = \kappa$.*

PROOF. Exercise. [*Hint.* Adapt the previous proof and use the other Löwenheim-Skolem theorem.] □

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[2]Why *elementary*? Well because there is also a notion of a diagram (without the word elementary), $\mathsf{Diag}(\mathcal{M})$, which is the same, but only for quantifier-free formulas.

[3]Actually, $\mathcal{N}$ is an elementary extension of a structure isomorphic to $\mathcal{M}$, but if we squint hard enough (and don't worry about formalities that can be handled) what I said in the main body of the text should be satisfactory.