Maths 410 – Homework 10

Due April 29, 2026 – Beginning of class.

Question 1. [20 marks] Let $f : \mathbb{R} \to \mathbb{R}$ and suppose that for all $x, y \in \mathbb{R}$ we have that:

$$|f(x) - f(y)| \leq (x - y)^2.$$

Prove that f is constant.

Question 2. [50 marks] Let $f : (a, b) \to \mathbb{R}$. Suppose that f is differentiable on (a, b). Suppose that $f'(x) > 0$ for all $x \in (a, b)$.

(1) [15 marks] Prove that f is strictly increasing on (a, b) [that is, for all $x, y \in (a, b)$, if $x < y$ then $f(x) < f(y)$.]

(2) [5 marks] Deduce that f is a bijection.

(3) [30 marks] Let $g = f^{-1}$ be the inverse function of f. Prove that g is differentiable and:

$$g'(f(x)) = \frac{1}{f'(x)},$$

for all $x \in (a, b)$.

Question 3.[30 marks] Suppose $g : (a, b) \to \mathbb{R}$ is differentiable and $g'(x)$ is bounded (i.e. there is some $M \in \mathbb{N}$ such that $g'(x) < M$ for all $x \in (a, b)$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Find a value $N \in \mathbb{R}$, depending on M such that if $\varepsilon < N$ then f is a bijection. [*Hint.* You may want to use Q.2(a)]

Extra Credit. Suppose that f' is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there is some $\delta > 0$ such that for all $x, y \in (a, b)$ if $0 < |x - y| < \delta$ then:

$$\left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < \varepsilon.$$