

## Maths 410 – Homework 11

Due May 12, 2026.

### I. Pointwise Convergence of Sequences of Functions

**Definition 1.** Let  $A$  be any set and  $(X, d)$  a metric space. For each  $n \in \mathbb{N}$  let  $f_n : A \rightarrow X$  be a function and let  $f : A \rightarrow X$  be a function. We say that  $(f_n)_{n \in \mathbb{N}}$  converges pointwise to  $f$ , denoted  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  if:

( $\star$ ) For all  $x \in A$ , we have that  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ .

*Remark* Equivalently, in the notation above,  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  if and only if:

( $\star'$ ) For all  $x \in A$  and all  $\varepsilon > 0$  there is some  $N \in \mathbb{N}$  (possibly depending on  $x$ ) such that for all  $n \geq N$  we have that  $f_n(x) \in B_\varepsilon(f(x))$ .

We will mainly be interested in the case where  $A \subseteq \mathbb{R}$  and  $(X, d)$  is  $\mathbb{R}$  with the Euclidean metric.

**Question 1** (*Warm-up*). For each  $n \in \mathbb{N}$  let  $f_n$  be the following function:

$$\begin{aligned} f_n : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto \frac{x^2 + nx}{n}. \end{aligned}$$

Prove that  $(f_n)_{n \in \mathbb{N}}$  converges pointwise to the function:

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R} \\ x &\mapsto x. \end{aligned}$$

**Question 2** (*The pointwise limit of continuous functions need not be continuous*). For each  $n \in \mathbb{N}$  let  $g_n$  be the following function:

$$\begin{aligned} g_n : [0, 1] &\rightarrow \mathbb{R} \\ x &\mapsto x^n. \end{aligned}$$

Find a function  $g : [0, 1] \rightarrow \mathbb{R}$  such that  $\lim_{n \rightarrow \infty} g_n(x) = g(x)$ . Deduce that the pointwise limit of continuous functions need not be continuous.

**Question 3** (*The pointwise limit of differentiable functions need not be differentiable*). For each  $n \in \mathbb{N}$  let  $h_n$  be the following function:

$$\begin{aligned} g_n : [-1, 1] &\rightarrow \mathbb{R} \\ x &\mapsto x^{1 + \frac{1}{2n-1}}. \end{aligned}$$

Show that  $(h_n)_{n \in \mathbb{N}}$  converges pointwise to the function:

$$h : [-1, 1] \rightarrow \mathbb{R} \\ x \mapsto |x|.$$

Deduce that the pointwise limit of differentiable functions need not be differentiable.

**Question 4** (*The pointwise limit of Riemann-integrable functions need not be Riemann-integrable*). Let  $\{r_0, r_1, \dots\}$  be a fixed enumeration of  $\mathbb{Q} \cap [0, 1]$ . For each  $n \in \mathbb{N}$  let  $\phi_n$  be the following function:

$$\phi_n : [0, 1] \rightarrow \mathbb{R} \\ x \mapsto \begin{cases} 1 & \text{if } x \in \{r_1, \dots, r_n\} \\ 0 & \text{otherwise.} \end{cases}.$$

Prove that for each  $n \in \mathbb{N}$  the function  $\phi_n$  is Riemann-integrable. Consider  $\lim_{n \rightarrow \infty} \phi_n(x)$ . Deduce that the pointwise limit of Riemann-integrable functions need not be Riemann-integrable.

The point of Questions 2-4 is that pointwise convergence of a sequence of functions does not preserve any of the notions we've been discussing so far in this course. Most of this will be remedied in the next part.

## II. Uniform Convergence of Sequences of Functions

**Definition 2.** Let  $A$  be any set and  $(X, d)$  a metric space. For each  $n \in \mathbb{N}$  let  $f_n : A \rightarrow X$  be a function and let  $f : A \rightarrow X$  be a function. We say that  $(f_n)_{n \in \mathbb{N}}$  converges uniformly to  $f$  (for short " $f_n \rightarrow f$  uniformly") if:

(†) For all  $\varepsilon > 0$  there is some  $N \in \mathbb{N}$  such that for all  $x \in A$  and all  $n \geq N$  we have that  $f_n(x) \in B_\varepsilon(f(x))$ .

*Remark.* It is immediate from the definitions that if  $(f_n)_{n \in \mathbb{N}}$  converges uniformly to some function  $f$  then it converges pointwise to the same function  $f$ . The next question shows that the converse is not true.

**Question 5** (*Warm-up*). Prove that the sequence  $(f_n)_{n \in \mathbb{N}}$  from Question 1 does not converge uniformly to  $f$ , on  $\mathbb{R}$ . Show that the convergence is uniform on any bounded subset of  $\mathbb{R}$ .

**Question 6** (*The uniform limit of continuous functions is continuous*). In the notation of Definition 2, suppose that  $A = (A, d_A)$  is a metric space and that each  $n \in \mathbb{N}$ , the function  $f_n : A \rightarrow X$  is continuous at some point  $a \in A$ . Show that if  $f_n \rightarrow f$  uniformly then  $f$  is also continuous at  $a \in A$ .

**Question 7** (*Uniform Convergence of Derivatives*). Let  $a < b \subset \mathbb{R}$  and for each  $n \in \mathbb{N}$  let  $f_n : [a, b] \rightarrow \mathbb{R}$  be a differentiable function. Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function such that  $(f_n)_{n \in \mathbb{N}}$  converges pointwise to  $f$ . Let  $g : [a, b] \rightarrow \mathbb{R}$  and suppose that  $f'_n \rightarrow g$  uniformly. Following the steps given below, show that  $f$  is differentiable and that  $f' = g$ .

*Main idea.* We want to show that for any  $c \in [a, b]$  the limit

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

exists and is equal to  $g(c)$ . To do this, it suffices to show the following:

( $\Downarrow$ ) For all  $\varepsilon > 0$  there is some  $\delta > 0$  such that for all  $x \in [a, b]$ :

$$\text{If } 0 < |x - c| < \delta \text{ then } \left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \varepsilon.$$

**Step 1.** Show that  $\left| \frac{f(x) - f(c)}{x - c} - g(c) \right|$  can be bounded above by:

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| + \left| \frac{f_n(x) - f_n(c)}{x - c} - f'_n(c) \right| + |f'_n(c) - g(c)|,$$

for any  $n \in \mathbb{N}$ .

**Step 2.** Show that for any  $\varepsilon > 0$  there is some  $N \in \mathbb{N}$  such that for all  $m, n \geq N$  we have that:

$$|f'_n(x) - f'_m(x)| < \frac{\varepsilon}{3} \text{ and } |f'_n(x) - g(c)| < \frac{\varepsilon}{3}$$

**Step 3.** For any  $N \in \mathbb{N}$  and any  $\varepsilon > 0$  show that there is some  $\delta > 0$  such that for all  $x \in [a, b]$ :

$$\text{If } 0 < |x - c| < \delta \text{ then } \left| \frac{f_N(x) - f_N(c)}{x - c} - f'_N(c) \right| < \frac{\varepsilon}{3}.$$

**Step 4.** The rest of the proof is dedicated to showing ( $\Downarrow$ ) with the  $\delta$  you found in Step 3, for the  $N$  you found in Step 2. All you need to show is that if  $0 < |x - c| < \delta$  then:

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_N(x) - f_N(c)}{x - c} \right| \leq \frac{\varepsilon}{3}.$$

To do this, apply the Mean Value Theorem to the function  $f_m - f_N$  (for  $m \geq N$ ) restricted to  $[x, c]$  (or  $[c, x]$ ) and then take lim as  $m \rightarrow \infty$ .

*Remark.* For a stronger version of the result you proved in Question 7, consult Theorem 7.17 from the textbook.

**Extra Credit.** Prove that the uniform limit of Riemann integrable functions is Riemann integrable.