Maths 410 – Homework 2

Due Feb. 16, 2026 – Beginning of class.

Question 1. [10 points] Let $\mathbb{I} := \mathbb{R} \setminus \mathbb{Q}$ denote the set of all irrational numbers. Explain why $\mathbb{I}$ is uncountable.

Question 2. [40 points] Follow the steps below to prove that the function:

$$\begin{aligned} f : \mathbb{N} \times \mathbb{N} &\to \mathbb{N} \\ (x, y) &\mapsto \frac{(x+y)(x+y+1)}{2} + y \end{aligned}$$

is a bijection.

- **Step 1.** Show that $f(x, y) = y + \sum_{k=1}^{x+y} k$. (You are allowed to use basic facts from Calculus 1/2 without proof.)
- **Step 2.** Deduce that if $x + y < x' + y'$ then $f(x, y) < f(x', y')$. Conclude that if $f(x, y) = f(x', y')$ then $x + y = x' + y'$. Conclude that f is injective.
- **Step 3.** Let $z \in \mathbb{N}$ be arbitrary, and show that if t_k is the largest integer of the form $\frac{k(k+1)}{2}$ (we call t_k the k-*th triangular number*) which does not exceed z, and $y = z - t_k$ and $x = k - y$, then $f(x, y) = z$. Conclude that the function is surjective.

[*Remark.* This completes the proof that $\mathbb{Q}$ is countable.]

Question 3. [20 points] Let $n \in \mathbb{N}_{>0}$. Construct a set of real numbers with exactly n limit points.

Question 4. [30 points] Let X be a set. Recall that the *discrete metric* is given by:

$$\begin{aligned} d : X \times X &\to \mathbb{R} \\ (x, y) &\mapsto \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

Prove that the discrete metric is a metric. Prove that for all $x \in X$, the set $\{x\}$ is open. Deduce that all subsets of X are both open and closed.

Extra Credit. A number $x \in \mathbb{R}$ is called *real algebraic* it is a root of a polynomial with integer coefficients. Prove that the set of all real algebraic numbers is countable.