Maths 410 – Homework 3

Due Feb. 23, 2026 – Beginning of class.

Question 1. [30 points] Let X be a metric space and $E \subseteq X$. Write E^o for the set of all interior points of E. We call E^o the *interior* of E.

(1) Prove that E^o is always open. Deduce that E is open if, and only if $E = E^o$.

(2) Prove that if $G \subseteq E$ and G is open, then $G \subseteq E^o$.

(3) Prove that the complement of E^o is the closure of the complement of E (i.e. prove that $X \setminus (E^o) = \overline{X \setminus E}$).

(4) Do E and $\overline{E}$ always have the same interior?

(5) Do E and E^o always have the same closures?

In the last two parts, justify your answers. [*Hint.* Consider $\mathbb{Q}^o$ and $\overline{\mathbb{Q}}$, as subsets of $\mathbb{R}$ with the Euclidean topology.]

Remark. Since arbitrary unions of open sets are open, parts (1) and (2) of this question show that E^o is the largest open subset contained in E.

Question 2. [10 points] Let X be an infinite set, equipped with the discrete metric (HW2). Prove that a subset of X is compact if and only if it is finite.

Question 3. [20 marks] Let $E = \{0\} \cup \left\{\frac{1}{n} : n \in \mathbb{N}_{>0}\right\} \subseteq \mathbb{R}$ (with the Euclidian metric). Show, directly from the definition, that E is compact.

Question 4. [40 marks] Let (X, d) be a metric space and $A, B \subseteq X$. We say that A and B are *separated* if $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. We say that E is *connected* if it cannot be written as the union of two non-empty separated sets.

(1) Show that if A and B are separated, then $A \cap B = \emptyset$. Show that the converse does not hold.

(2) Show that if A and B are disjoint and closed, then they are separated.

(3) Show that if A and B are disjoint open sets they are separated. Deduce that for any $x \in X$ and any $r > 0$, the sets $B_r(x)$ and $\{y \in X : d(x, y) > r\}$ are separated.

Extra Credit. Show that if X is a connected metric space (i.e. X is connected) and contains at least two points, then X is uncountable. [*Hint.* Use 4(c).]