Maths 410 – Homework 4

Due March 2, 2026 – Beginning of class.

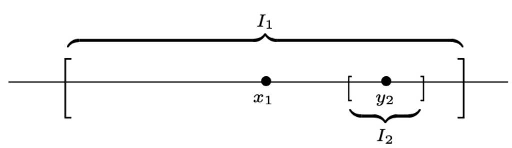
Definition. Let (X, d) be a metric space and $E \subseteq X$. We say that $x \in E$ is an *isolated point of* E if it is not a limit point of E. We say that E is *perfect* if E is closed and contains no isolated points.

Question 1. [50 points] The goal of this question is to prove that every non-empty perfect subset of $\mathbb{R}$ is uncountable. Let $P \subseteq \mathbb{R}$ be a non-empty perfect set.

(1) Carefully prove that P is infinite.

(2) Suppose towards a contradiction that $P = \{x_n : n \in \mathbb{N}\}$ is countable. Let $I_1 \subseteq \mathbb{R}$ be a closed and bounded interval such that $x_1 \in I_1^o$. Prove that there is some $y_2 \in P \setminus \{x_1\}$ such that y_2 is an interior point of I_1. [Here you need to use that x_1 is not isolated.]

(3) Suppose that $I_1 = [a, b]$, for $a < b \in \mathbb{R}$, Construct a closed interval $I_2 \subseteq I_1$ such that $y_2 \in I_2$ but $x_1 \notin I_2$. *Hint.* Consider the following picture:

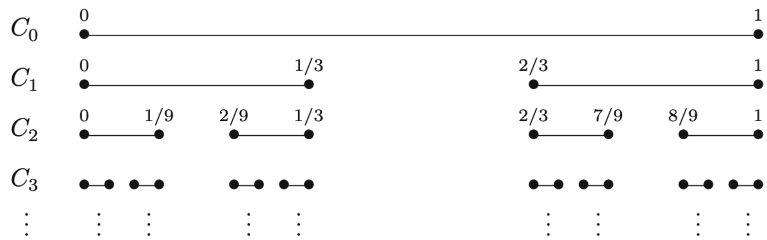


(4) Explain why we may continue this process so that for all $n \in \mathbb{N}$ we can find a closed interval I_n such that $I_{n+1} \subseteq I_n$, $x_n \notin I_{n+1}$, and $I_n \cap P \neq \emptyset$. [*Hint.* The idea is that we may always insist that $y_{n+1} \neq x_n$.]

(5) Let $K_n = I_n \cap P$. Show that $\bigcap_{n \in \mathbb{N}} K_n = \emptyset$ and derive a contradiction.

Question 2. [50 points] The goal of this question is to build an uncountable subset of $\mathbb{R}$ which contains no open intervals. The set C we will construct is known as the *Cantor Set*, and the construction goes as follows: Let $C_0 = [0, 1]$. Let C_1 be the subset of C_0 obtained by removing the open interval in the "middle third", i.e $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Let C_2 be obtained by removing the middle third open interval from each of the two intervals of C_1 (see picture). Continuing like this gives us a sequence $C_0 \supseteq C_1 \cdots \supseteq C_n \supseteq \cdots$, where each C_n consists of 2^n disjoint closed intervals, each having length $\frac{1}{3^n}$.

Here's the picture:



(1) Prove that C_n is compact, for each $n \in \mathbb{N}$. Deduce that:
$$C := \bigcap_{n \in \mathbb{N}} C_n$$
is non-empty. [*Remark.* In fact, it is clear that C is non-empty, as it contains all the endpoints of the intervals that appear in the construction.]

(2) Explain why, for all $k, m \in \mathbb{N}$ we have that:
$$C \cap \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}, \right) = \emptyset.$$
Deduce that for all $a < b \in \mathbb{R}$ we have that (a, b) is not a subset of C.

(3) Let $x \in C$. Prove that x is not isolated. Deduce that C is perfect. Using Question 1, explain why C contains more points than the endpoints of the intervals of C_n.

Extra Credit. Construct a perfect set $P \subseteq \mathbb{R}$ such that $P \cap \mathbb{Q} = \emptyset$.

[*Hint.* Redo the construction of the Cantor set, but with irrational endpoints. Fix an enumeration of the rationals within the endpoints you picked, and at stage n remove make sure to remove the n-th rational number in your enumeration.]