Maths 410 – Homework 5

Due March 9, 2026 – Beginning of class.

Question 1. [40 points] Let $(x_n)_{n\in\mathbb{N}}$ be a real sequence such that $\lim_{n\to\infty} x_n = x$. For each $n \in \mathbb{N}$ let $s_n := \frac{1}{n}\sum_{i=1}^{n} x_n$ Show that $\lim_{n\to\infty} s_n = x$. Is the converse [If $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_n = x$, then $\lim_{n\to\infty} x_n = x$] true?

Question 2. [30 points] Show that $(x_n)_{n\in\mathbb{N}}$ converges to x if, and only if, every subsequence of $(x_n)_{n\in\mathbb{N}}$ converges to x.

Question 3. [30 marks] Let $(x_n)_{n\in\mathbb{N}}$ be a real sequence. Prove that if $(x_n)_{n\in\mathbb{N}}$ converges then $(|x_n|)_{n\in\mathbb{N}}$ converges. Is the converse true?

Extra Credit. Define a sequence inductively, as follows: $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + \sqrt{x_n}}$. Show that this sequence is increasing and bounded above.