Maths 410 – Homework 6

Due April 1, 2026 – Beginning of class.

Question 1. [40 points] Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in some metric space X (not necessarily a complete space). Prove that $(x_n)_{n\in\mathbb{N}}$ converges if and only if it has *some* convergent subsequence.

Question 2. [40 points] Prove that if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are Cauchy sequence in some metric space X, then, the real sequence $(d(x_n, y_n))_{n\in\mathbb{N}}$ converges. [*Hint.* It is easier to prove that the sequence in question is Cauchy.]

Question 3. [20 points] Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence of non-negative terms. Suppose that $\sum_{n=1}^{\infty} a_n$ diverges. Prove that $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ diverges.

Extra Credit. Let X be a complete metric space, and for each $n \in \mathbb{N}$ let $E_n \subseteq X$ be non-empty, closed, and bounded. Suppose that $E_0 \supseteq E_1 \supseteq E_2 \supseteq \cdots$ and

$$\lim_{n\to\infty} \mathsf{diam}(E_n) = 0.$$

Prove that, then, $\bigcap_{n\in\mathbb{N}} E_n$ consists of exactly one point.