Maths 410 – Homework 7

Due April 8, 2026 – Beginning of class.

Question 1. [35 points] Consider the following series:

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}.$$

Prove that the series converges if $p > 1$ and diverges if $p \leq 1$. You may use without proof the fact that $\log : \mathbb{R}_{>0} \to \mathbb{R}$ is an increasing function. [*Hint.* Use Cauchy's theorem.]

Question 2. [30 points] Follow the steps given below to prove that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is irrational.

(1) Suppose that e is rational. Write $e = \frac{p}{q}$ and deduce that $0 < q!(e - s_q) < \frac{1}{q}$, where s_q is the q-th partial sum of the series $\sum_{n=0}^{\infty} \frac{1}{n!}$.

(2) Prove that $q!s_q$ is an integer.

(3) Deduce that $q!(e - s_q)$ is an integer, and derive a contradiction.

Question 3. [35 points] Let $f : \mathbb{R} \to \mathbb{R}$. Suppose that for all $x \in \mathbb{R}$ we have that:

$$\lim_{\varepsilon \to 0} [f(x + \varepsilon) - f(x - \varepsilon)] = 0$$

Does this imply that f is continuous? [*Hint.* Consider the function f sending x to 1 if x is an integer and to 0 otherwise.]

Extra Credit. Given a sequence $\alpha \in \{0, 2\}^{\mathbb{N}}$ (i.e. a sequence of 0's and 2's), prove that:

$$s(\alpha) = \sum_{n=1}^{\infty} \frac{\alpha(n)}{3^n}$$

converges. Let $S = \{S(\alpha) : \alpha \in \{0, 2\}^{\mathbb{N}}\}$. Prove that S is precisely the Cantor set (from HW4).