

Maths 410 – Homework 8

Due April 15, 2026 – Beginning of class.

Question 1. [30 points]

- (1) [20 points] Let X, Y be metric spaces and $f : X \rightarrow Y$ a continuous map. Let $E \subseteq X$. Prove that:

$$f(\overline{E}) \subseteq \overline{f(E)}.$$

In the next sub-question, you are asked to show that the inclusion can be strict.

- (2) [10 points] Consider the following function:

$$f : \mathbb{R} \rightarrow \mathbb{R} \\ x \mapsto \frac{x}{1+x^2}.$$

Let $E = [1, \infty) \subseteq \mathbb{R}$. Prove that $f(\overline{E}) = (0, \frac{1}{2}]$. Conclude that $f(\overline{E})$ is a proper subset of $\overline{f(E)}$.

Question 2. [10 points] Let $\phi : X \rightarrow \mathbb{R}$ be a continuous real-valued function. Let $Z(\phi) := \{x \in X : \phi(x) = 0\}$. Prove that $Z(\phi)$ is a closed subset of X .

Question 3. [40 points] Let X, Y be metric spaces and $D \subseteq X$ a dense subset of X . Let $f, g : X \rightarrow Y$ be continuous maps and suppose that for all $p \in D$ we have that $f(p) = g(p)$.

- (1) [5 points] Prove that $f(D)$ is dense in $f(X)$. [*Hint.* Use Question 1.]
- (2) [20 points] Prove that the function:

$$\phi : X \rightarrow \mathbb{R} \\ x \mapsto d_Y(f(x), g(x))$$

is continuous.

- (3) [15 points] Conclude, using the previous parts and Question 2 (or otherwise), that $f(x) = g(x)$ for all $x \in X$.

Question 4. [20 points] Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function and assume that f is uniformly continuous on a bounded subset $E \subseteq \mathbb{R}$. Show that f is bounded on E . Does the result conclusion still hold if we don't assume that E is bounded?

Extra Credit. Let $I = [0, 1] \subseteq \mathbb{R}$. Prove that if $f : I \rightarrow I$ is continuous, then there is some $x \in I$ such that $f(x) = x$.