Maths 410 – Homework 9

Due April 22, 2026 – Beginning of class.

The goal of this question is to prove the following theorem:

Theorem. Let X be a metric space and A, B non-empty disjoint closed subsets of X. Then, there exist disjoint open sets U, V such that $A \subseteq X$ and $B \subseteq V$.

Question 1. [30 points] Let (X, d) be a metric space and fix a non-empty subset $E \subseteq X$. Define the function:

$$\begin{aligned} \rho_E : X &\to \mathbb{R} \\ x &\mapsto \inf_{z \in E} d(x, z) \end{aligned}$$

(1) [5 points] Briefly explain why ρ_E is well defined.

(2) [10 points] Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.

(3) [10 points] Prove that for all $x, y \in X$ we have:

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

(4) [5 points] Deduce that ρ is uniformly continuous on X.

Question 2. [35 points] Let $K, F \subseteq X$ be disjoint subsets of the metric space (X, d). Suppose that K is compact and F is closed. Prove that there is some $\delta > 0$ such that for all $x \in K$ and all $y \in F$ we have that

$$d(x, y) > \delta.$$

Hint. You may want to use that ρ_F is a positive continuous function on the compact set K.

Extra Credit (1). Does the result in Question 2 still hold if we only assume that K is closed? Justify your answers.

Question 3. [35 points] Let A, B be disjoint non-empty closed subsets of X. Consider the function:

$$\begin{aligned} f : X &\to \mathbb{R} \\ x &\mapsto \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}. \end{aligned}$$

Show that f is continuous and $f(X) \subseteq [0, 1]$.

Extra Credit (2). Using the function f constructed in Question 3, or otherwise, finish the proof of the theorem.