Maths 410: Real Analysis

Midterm

Friday, March 13 2026 – 12:00-1:00pm

This exam consists of **3 + 1 questions**. The fourth question is **Extra Credit**

Good luck :)

Question	Marks
1	/40
2	/40
3	/20
4	/20
Total	/100

Question 1

Answer the following True or False questions. You do not need to justify your answers. A correct answer is worth 5 marks while an incorrect answer carries a penalty of -5 marks.

Therefore, it may be preferable to leave some answers blank!

(a) There is a metric space (X, d) all of whose subsets are compact.
True: Take a finite metric space with the discrete metric.

(b) There is an infinite metric space all of whose subsets are open.
True: Take any infinite metric space, again, with the discrete metric.

(c) If (X, d) is a metric space, $Y \subseteq X$ and $(y_n)_{n\in\mathbb{N}}$ is a sequence in Y which converges in X then it converges in Y.
False: The sequence $(\frac{1}{n})_{n\in\mathbb{N}_{>1}}$ converges in $\mathbb{R}$ but does not converge in $(0, 1]$.

(d) A subset K of a metric space (X, d) is compact if and only if it is closed and bounded.
False: This is not true in an infinite metric space with the discrete metric.

(e) If $a, b \in \mathbb{R}$ are irrational then so is at least one of $a + b$ or $a \times b$.
False: If $a = \sqrt{2}, b = -\sqrt{2}$ then $a + b = 0$ and $a \times b = 2$.

(f) The union of infinitely many compact subsets of a metric space is compact.
False: In $\mathbb{R}$, each interval $[-n, n]$, for $n \in \mathbb{N}_{>0}$ is compact (by Heine-Borel) but $\bigcup_{n\in\mathbb{N}_{>0}}[-n, n] = \mathbb{R}$ is not compact.

(g) Let (X, d) be a metric space and $Y \subseteq X$. If $K \subseteq Y$ is compact in the metric space $(Y, d \restriction_Y)$ then it is compact in X.
True: We have proved this in class.

(h) There is a perfect set $P \subseteq \mathbb{R}$ (with the Euclidean metric) such that $P \subseteq \mathbb{Q}$.
False: Perfect sets are uncountable.

Question 2

Let (X, d) be a metric space.

(a) [5 marks] Let $Y \subseteq X$. Define what it means for a point $x \in X$ to be a limit point of Y. Define the closure $\overline{Y}$ of Y.

(b) [5 marks] Let $Y \subseteq X$. Define what it means for a point $x \in X$ to be an isolated point of Y.

(c) [20 marks] Let $Y = \{y_1, \dots, y_n, \dots\} \subseteq X$ be a set and assume that there is some point $a \in X$ such that $d(a, y_n) < \frac{1}{n}$, for each $n \in \mathbb{N}_{>0}$. Show that Y has exactly one limit point.

(d) [10 marks] Prove that if (X, d) is an infinite metric space, then it contains an infinite subset $Y \subseteq X$ such that every point of Y is isolated in Y.

Answer

(a) A point $x \in X$ is a *limit point* of $Y \subseteq X$ if for all $\varepsilon > 0$ the set $(B_\varepsilon(x) \cap Y) \setminus \{x\}$ is non-empty. The *closure* of Y is the set $Y \cup Y'$, where Y' is the set of all limit points of Y.

(b) We say that $x \in X$ is an *isolated point* of Y if $x \in Y$ and x is not a limit point of Y.

(c) We claim that $Y' = \{a\}$. First, we show that $\{a\} \subseteq Y'$. Let $\varepsilon > 0$. By the Archimedean property of $\mathbb{R}$ there is some $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$. Then, for all $n \geq N$ we have that $d(a, x_n) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ so, for all $n \geq N$ we have that $x_n \in B_\varepsilon(a)$. Thus, $(B_\varepsilon(a) \cap Y) \setminus \{a\} \neq \emptyset$. Conversely, suppose that $a' \in X$ is such that $a' \in Y'$. Let $\varepsilon > 0$. We will show that $d(a, a') < \varepsilon$. Since a' is a limit point of Y there are infinitely many $m \in \mathbb{N}$ such that $d(x_m, a') < \frac{\varepsilon}{2}$. Arguing as before, there is also some $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $x_n \in B_{\frac{\varepsilon}{2}}(a)$. In particular, there is some $m \in \mathbb{N}$ such that $d(x_m, a') < \frac{\varepsilon}{2}$ and $d(x_m, a) < \frac{\varepsilon}{2}$, but then $d(a, a') < d(a, x) + d(x, a') < \varepsilon$, as required.

(d) If X has no non-isolated points, then we are done. So suppose that X has at least one point $a \in X$ which is non-isolated. Since a is non-isolated in X, for each $n \in \mathbb{N}_{>1}$ there are infinitely many points $y \in B_{\frac{1}{n}}(a) \setminus \{a\}$. Inductively construct a set $Y = \{y_n : n \in \mathbb{N}\}$ so that for each $n \in \mathbb{N}_{>1}$ we have that $y_n \in B_{\frac{1}{n}}(a) \setminus \{a\}$ and $y_n \neq y_i$, for $i < n$. By construction, Y is infinite, and by (c), the only limit point of Y is a, hence every point of Y is isolated.

Question 3

(a) [5 marks] Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d) and $x \in X$. Define what $\lim_{n\to\infty} x_n = x$ means.

(b) [5 marks] Let X be a set. Define the *discrete metric* on X.

(c) [10 marks] A sequence $(x_n)_{n\in\mathbb{N}}$ in a metric space (X, d) is called *eventually constant* if there is some $x \in X$ and some $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $x_n = x$. Prove that if (X, d) is a metric space with the discrete metric then a sequence $(x_n)_{n\in\mathbb{N}}$ converges if and only if it is eventually constant.

Answer

(a) We say that $\lim_{n\to\infty} x_n = x$ if, for all $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $d(x_n, x) < \varepsilon$.

(b) The discrete metric on a set X is defined as follows:

$$\begin{aligned} d : X \times X &\to \mathbb{R} \\ (x, y) &\mapsto \begin{cases} 0 & \text{if} x = y \\ 1 & \text{otherwise.} \end{cases} \end{aligned}$$

(c) If $(x_n)_{n\in\mathbb{N}}$ is eventually constant, then it converges (in any metric space). Conversely, suppose that $(x_n)_{n\in\mathbb{N}}$ is a sequence in a metric space (X, d) with the discrete metric and $\lim_{n\to\infty} = x$, for some $x \in X$. Then, for $\varepsilon = \frac{1}{2}$ there is some $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $d(x_n, x) < \frac{1}{2}$. In particular, for all $n \geq N$ we have that $x_n = x$, so $(x_n)_{n\in\mathbb{N}}$ is eventually constant.

Question 4 – Extra Credit

Let $X \subseteq \mathbb{R}$ be any closed set. Prove that there is a sequence $(x_n)_{n\in\mathbb{N}}$ such that the set of all subsequential limits of $(x_n)_{n\in\mathbb{N}}$ is X.

Answer