

Maths 410: Real Analysis

Midterm

Friday, March 13 2026 – 12:00-1:00pm

This exam consists of **3 + 1** questions. The fourth question is **Extra Credit**

Good luck :)

Question	Marks
1	/40
2	/40
3	/20
4	/20
Total	/100

Question 1

Answer the following True or False questions. You do not need to justify your answers. A correct answer is worth 5 marks while an incorrect answer carries a penalty of -5 marks.

Therefore, it may be preferable to leave some answers blank!

- (a) _____: There is a metric space (X, d) all of whose subsets are compact.
- (b) _____: There is an infinite metric space all of whose subsets are open.
- (c) _____: If (X, d) is a metric space, $Y \subseteq X$ and $(y_n)_{n \in \mathbb{N}}$ is a sequence in Y which converges in X then it converges in Y .
- (d) _____: A subset K of a metric space (X, d) is compact if and only if it is closed and bounded.
- (e) _____: If $a, b \in \mathbb{R}$ are irrational then so is at least one of $a + b$ or $a \times b$.
- (f) _____: The union of infinitely many compact subsets of a metric space is compact.
- (g) _____: Let (X, d) be a metric space and $Y \subseteq X$. If $K \subseteq Y$ is compact in the metric space $(Y, d|_Y)$ then it is compact in X .
- (h) _____: There is a perfect set $P \subseteq \mathbb{R}$ such that $P \subseteq \mathbb{Q}$.

Question 2

Let (X, d) be a metric space.

- (a) [5 marks] Let $Y \subseteq X$. Define what it means for a point $x \in X$ to be a limit point of Y . Define the closure \bar{Y} of Y .
- (b) [5 marks] Let $Y \subseteq X$. Define what it means for a point $x \in X$ to be an isolated point of Y .
- (c) [20 marks] Let $Y = \{y_1, \dots, y_n, \dots\} \subseteq X$ be a set and assume that there is some point $a \in X$ such that $d(a, y_n) < \frac{1}{n}$, for each $n \in \mathbb{N}_{>0}$. Show that Y has exactly one limit point.
- (d) [10 marks] Prove that if (X, d) is an infinite metric space, then it contains an infinite subset $Y \subseteq X$ such that every point of Y is isolated in Y .

Answer

(a)

(b)

(c)

(d)

Question 3

- (a) [5 marks] Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in a metric space (X, d) and $x \in X$. Define what $\lim_{n \rightarrow \infty} x_n = x$ means.
- (b) [5 marks] Let X be a set. Define the *discrete metric* on X .
- (c) [10 marks] A sequence $(x_n)_{n \in \mathbb{N}}$ in a metric space (X, d) is called *eventually constant* if there is some $x \in X$ and some $N \in \mathbb{N}$ such that for all $n \geq N$ we have that $x_n = x$. Prove that if (X, d) is a metric space with the discrete metric then a sequence $(x_n)_{n \in \mathbb{N}}$ converges if and only if it is eventually constant.

Answer

(a)

(b)

(c)

Question 4 – Extra Credit

Let $X \subseteq \mathbb{R}$ be any closed set. Prove that there is a sequence $(x_n)_{n \in \mathbb{N}}$ such that the set of all subsequential limits of $(x_n)_{n \in \mathbb{N}}$ is X .

Answer