Maths 410: Real Analysis

Practice Final

2 Hours

This exam consists of **4 + 1 questions**. The real final will have a fifth question which will be **Extra Credit**.

Good luck :)

Question	Marks
1	/40
2	/20
3	/20
4	/20
Total	/100

Question 1

Answer the following True or False questions. You do not need to justify your answers. A correct answer is worth 5 marks while an incorrect answer carries a penalty of -3 marks. The minimum grade you can get in this question is 0.

(a) _____: Every Cauchy sequence of real numbers is bounded.

(b) _____: Every bounded sequence of real numbers converges.

(c) _____: If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

(d) _____: The image of a closed set under a continuous function is closed.

(e) _____: Every continuous function $f : [0, \infty) \to \mathbb{R}$ is uniformly continuous.

(f) _____: If $f_n \to f$ uniformly and each f_n is continuous, then f is continuous.

(g) _____: The composition of two Riemann integrable functions is Riemann integrable.

(h) _____: There is a function $f : [a, b] \to \mathbb{R}$ with infinitely many discontinuities which is Riemann integrable.

Question 2 – Continuity

(a) [5 marks] Define what it means for a function to be:

(i) continuous at a point;

(ii) uniformly continuous on a set.

(b) [7 marks] Prove that if $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded.

(c) [8 marks] Let $f : (0, 1) \to \mathbb{R}$ be uniformly continuous. Prove that f is bounded on $(0, 1)$.

Answer

(a)

(b)

(c)

Question 3 – Differentiation

(a) [5 marks] State the Mean Value Theorem.

(b) [7 marks] Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b), with $f'(x) = 0$ for all $x \in (a, b)$. Prove that f is constant on $[a, b]$.

(c) [8 marks] Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and suppose that $|f'(x)| \leq M$ for all $x \in \mathbb{R}$. Prove that f is uniformly continuous on $\mathbb{R}$.

Answer

(a)

(b)

(c)

Question 4 – Integration

(a) [5 marks] Define what it means for a bounded function $f : [a, b] \to \mathbb{R}$ to be Riemann integrable.

(b) [7 marks] Carefully state the Fundamental Theorem of Calculus.

(c) [8 marks] Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Prove that there exists $c \in [a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

Answer

(a)

(b)

(c)