Maths 410: Real Analysis

Practice Midterm

This exam is meant to help you practice for the real midterm, so you're meant to be able to solve it in 60 minutes.

This exam consists of **3 + 1 questions**. The fourth question is **Extra Credit**

Good luck :)

Question	Marks
1	/40
2	/40
3	/20
4	/20
Total	/100

Question 1

Answer the following True or False questions. You do not need to justify your answers. A correct answer is worth 5 mariks and a wrong answer is is worth -5 marks. **Therefore, it may be preferable to leave some answers blank!**

(a) ____________: If (X, d_X) and (Y, d_Y) are metric spaces then $(X \times Y, d)$, where $d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2)$ is a metric space.

(b) ____________: There is a compact metric space all of whose subsets are open.

(c) ____________: If (X, d) is a metric space, $Y \subseteq X$ and $V \subseteq Y$ is closed in Y then it is closed in X.

(d) ____________: A subset K of a metric space (X, d) is compact if and only if whenever $E \subseteq K$ is infinite then E has a limit point in X.

(e) ____________: Let A, B be uncountable subsets of a set X. If $A \cap B$ is infinite, then $A \cap B$ is uncountable.

(f) ____________: The union of infinitely many compact subsets of a metric space is closed.

(g) ____________: Let (X, d) be a metric space and $Y \subseteq X$. If $K \subseteq Y$ is compact in the metric space $(Y, d \restriction_Y)$ then it is compact in X.

(h) ____________: Let $A, B \subseteq X$ be connected subsets. Suppose that $A \cap B \neq \emptyset$. Then $A \cup B$ is connected.

Question 2

Let (X, d) be a metric space.

(a) [10 marks] Let $x, y \in X$ and $\varepsilon > 0$. Prove that if $y \in B_{\frac{\varepsilon}{2}}(x)$, then $B_{\frac{\varepsilon}{2}}(y) \subseteq B_{\varepsilon}(x)$.

(b) [30 marks] Given $A \subseteq X$ and $x \in X$ define:

$$d(x, A) := \inf\{d(x, a) : a \in A\}.$$

Prove that $x \in \overline{A}$ if and only if $d(x, A) = 0$.

Question 3

(a) [5 marks] Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d) and $x \in X$. Define what $\lim_{n\to\infty} x_n = x$ means.

(b) [5 marks] Show, directly from the definition, that if $(x_n)_{n\in\mathbb{N}}$ converges then it is bounded.

(b) [10 marks] Show, directly from the definition, that a sequence $(x_n)_{n\in\mathbb{N}}$ cannot have more than one limit.

Question 4 – Extra Credit

Let $X = \{0,1\}^{\mathbb{N}}$ (i.e. infinite sequences of 0s and 1s) and define a function:

$$\begin{aligned} d : X \times X &\to \mathbb{R}_{\geq 0} \\ ((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}) &\mapsto \begin{cases} 0 & \text{if } (x_n)_{n\in\mathbb{N}} = (y_n)_{n\in\mathbb{N}}; \\ \frac{1}{\min\{n\in\mathbb{N} : x_n \neq y_n\}+1} & \text{otherwise.} \end{cases} \end{aligned}$$

Prove that (X, d) is a metric space.