

**Theorem 1** (Chain Rule). *Let  $f : [a, b] \rightarrow \mathbb{R}$ ,  $g : [c, d] \rightarrow \mathbb{R}$  and assume that  $f([a, b]) \subseteq [c, d]$ . Let  $p \in [a, b]$ . If  $f$  is differentiable at  $p$  and  $g$  is differentiable at  $f(p)$ , then  $g \circ f : [a, b] \rightarrow \mathbb{R}$  is differentiable at  $p$ . More precisely:*

$$(g \circ f)'(p) = g'(f(p))f'(p).$$

*Proof.* By assumption, the following limit:

$$\lim_{y \rightarrow f(p)} \frac{g(y) - g(f(p))}{y - f(p)}$$

exists, and it is, of course, equal to  $g'(f(p))$ . Define a new function:

$$Q : [c, d] \rightarrow \mathbb{R}$$

$$y \mapsto \begin{cases} \frac{g(y) - g(f(p))}{y - f(p)} & \text{if } y \neq f(p) \\ g'(f(p)) & \text{otherwise.} \end{cases}$$

Then,  $Q$  is a well-defined function, and it is continuous on  $[c, d]$ . For all  $y \in [c, d]$  we have that:

$$Q(y)(y - f(p)) = g(y) - g(f(p)).$$

Indeed, this is by definition when  $y \neq f(p)$  and, when  $y = f(p)$  we simply have

$$Q(f(p))(f(p) - f(p)) = 0 = g(f(p)) - g(f(p)).$$

In particular, for any  $x \in [a, b]$  we have that:

$$Q(f(x))(f(x) - f(p)) = g(f(x)) - g(f(p)).$$

Now, it follows from the above that, if  $x \neq p$  then we have that:

$$Q(f(x)) \left( \frac{f(x) - f(p)}{x - p} \right) = \frac{g(f(x)) - g(f(p))}{x - p}.$$

Taking the limit as  $x \rightarrow p$  on both sides we have that:

$$\lim_{x \rightarrow p} \left[ Q(f(x)) \left( \frac{f(x) - f(p)}{x - p} \right) \right] = \lim_{x \rightarrow p} \left[ \frac{g(f(x)) - g(f(p))}{x - p} \right].$$

The limit on the RHS (if it exists) is precisely  $(g \circ f)'(p)$ . To show that this limit exists, it suffices to show that the limit on the LHS exists. Because  $f$  is continuous at  $p$  and  $Q$  is continuous at  $f(p)$ , we have that  $\lim_{x \rightarrow p} Q(f(x)) = Q(f(p)) = g'(f(p))$ . Since  $f$  is differentiable at  $p$  the limit  $\lim_{x \rightarrow p} \frac{f(x) - f(p)}{x - p}$  exists and is equal to  $f'(p)$ . Thus, both limits in the product in the LHS exist, and hence the limit in the RHS exists and is equal to their product.  $\square$