Maths 410 – Extra Notes (Week 4)

Throughout, let (X, d) be a metric space. If $Y \subseteq X$, then $d' = d \restriction_{Y \times Y}$ is a metric on Y,[1] so we may view (Y, d') as a metric space. To simplify notation, if $y_1, y_2 \in Y$ we write $d(y_1, y_2)$ for $d'(y_1, y_2)$. Morally, if we know how far away any two elements of X are and $Y \subseteq X$, then we know how far away any two elements of Y are!

Recall. For $x \in X$ and $r \in \mathbb{R}_{>0}$ we write:

$$B_r(x) := \{z \in X : d(x, z) < r\},$$

for the *open ball of radius* r *about* x. This notation can be a bit imprecise when we are working with two metric spaces, but we can remedy this, by specifying in which metric space we're working. For example:

$$B_r^{X}(x) := \{z \in X : d(x, z) < r\}.$$

So, if $Y \subseteq X$ and $y \in Y$, we can also consider:

$$B_r^{Y}(y) := \{z \in Y : d(z, y) < r\}.$$

In this notation, we have:

$$B_r^Y(y) = B_r^X(y) \cap Y.$$

Let's now remind ourselves of the definition of an interior point:

Definition. Let $E \subseteq X$. We say that $p \in E$ is an *interior point of* E *(in* X*)* if there is some $r \in \mathbb{R}_{>0}$ such that $B_r^X(x) \subseteq E$.

So if $x \in E \subseteq Y \subseteq X$ there are two "competing" notions of x being an interior point of E:

- x could be an interior point of E in X, meaning that there is some $r \in \mathbb{R}_{>0}$ such that $B_r^{X}(x) \subseteq E$.
- x could be an interior point of E in Y, meaning that there is some $r \in \mathbb{R}_{>0}$ such that $B_r^{Y}(x) \subseteq E$.

In this notation:

- If x is an interior point of E in X then it is an interior point of E in Y

[1] Let $f : A \to B$ be a function and $A' \subseteq A$. The notation $f \restriction_{A'}$ means the *restriction* of f to A', i.e. the function $f' : A' \to B$ such that $f'(a) = f(a)$ for all $a \in A'$.

Proof. By assumption, for each $x \in E$ there is some $r \in \mathbb{R}_{>0}$ such that $B_r^X(x) \subseteq E$. But then:

$$B_r^Y(x) = B_r^X(x) \cap Y \subseteq E \cap = E,$$

so x is an interior point of E in Y. $\square$

- If x is an interior point of E in Y, then x NEED NOT be an interior point of E in X.

Examples. As usual, $\mathbb{R}$ is taken with the Euclidean metric.

1. Let $X = \mathbb{R}$, $Y = \mathbb{Q}$ and $E \subseteq Y$ the following set:

$$E = \left\{ \frac{x}{y} : x, y \in \mathbb{Z}, 0 \leq |x| < |y| \right\}.$$

Then, $E = (-1, 1) \cap \mathbb{Q} = B_1^{\mathbb{Q}}(0)$, so E is open in Y (it is an open ball), but it is not open in $\mathbb{R}$ (in fact, $E^o = \emptyset$).

2. Let $X = \mathbb{R}$ and take Y to be the following set:

$$Y = \bigcup_{m \in \mathbb{N}} \left\{ m + \frac{1}{n} : n \in \mathbb{N}_{>0} \right\} = \left\{ \frac{1}{2}, \frac{1}{3}, \cdots \right\} \cup \left\{ \frac{3}{2}, \frac{4}{3}, \cdots \right\} \cup \cdots.$$

Consider the set:

$$E = \left\{ \frac{1}{n} : n \in \mathbb{N}_{>1} \right\} = \left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots \right\}.$$

Then, $E = B_{\frac{1}{2}}^Y\left(\frac{1}{2}\right)$, and is thus open in Y but, again, it has no interior points, and is thus not open in X.

When $Y \subseteq X$, the relation of the open subsets of Y and the open subsets of X is given precisely by the following theorem:

Theorem. Let $E \subseteq Y \subseteq X$. Then, the following are equivalent:

1. E is open in Y.
2. There is some set $G \subseteq X$ which is open in X such that $E = Y \cap G$.