Alternative proof that the limit of the product of convergent sequences is the product of their limits.

The proof from lectures that if $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are real sequences such that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y,$$

then

$$\lim_{n\to\infty} x_n y_n = xy$$

used a mysterious identity. Here I'll give you a more "systematic" (but longer) proof.

Case 1. Suppose that $x \neq 0$.

Let $\varepsilon > 0$. We want to show that there is some $N \in \mathbb{N}$ such that if $n \geq N$ then $|x_n y_n - xy| < \varepsilon$.

We have that:

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n + (xy_n - xy_n) - xy| \\
&= |x_n y_n - xy_n + xy_n - xy| \\
&\leq |x_n y_n - xy_n| + |xy_n - xy| \\
&= |y_n||x_n - x| + |x||y_n - y|
\end{aligned}$$

- By assumption, there is some $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then

  $$|y_n - y| < \frac{\varepsilon}{2|x|},$$

  [Here we use that $x \neq 0$.]

- Since $(y_n)_{n\in\mathbb{N}}$ converges, it is bounded. Hence, there is some $M \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have that $|y_n| < M$. Without loss of generality, we may assume that $M \geq 1$. Let $N_2 \in \mathbb{N}$ be such that for all $n \geq N_2$ we have that:

  $$|x_n - x| < \frac{\varepsilon}{2M}.$$

If $N = \max\{N_1, N_2\}$, then, for all $n \geq N$ we have that:

$$\begin{aligned} |x_n y_n - xy| &\leq |y_n||x_n - x| + |x||y_n - y| \\ &< |y_n|\frac{\varepsilon}{2M} + |x|\frac{\varepsilon}{2|x|} \\ &\leq M\frac{\varepsilon}{2M} + |x|\frac{\varepsilon}{2|x|} = \varepsilon. \end{aligned}$$

Case 2. Suppose that $x = 0$.

Claim. Suppose that $(y_n)_{n\in\mathbb{N}}$ is bounded (not necessarily convergent) and $\lim_{n\to\infty} x_n = 0$. Then, $\lim_{n\to\infty} x_n y_n = 0$.

Proof of Claim Let $M \in \mathbb{N}$ be such that $|y_n| \leq M$ for all $n \in \mathbb{N}$. Without loss of generality, we may assume that $M > 0$.

Given $\varepsilon > 0$ we want to show that there is some $N \in \mathbb{N}$ such that whenever $n \geq N$ then $|x_n y_n| < \varepsilon$. By assumption, there is some $N \in \mathbb{N}$ such that if $n \geq N$ then $|x_n| < \frac{\varepsilon}{M}$. Then $|x_n y_n| \leq M|x_n| < M\frac{\varepsilon}{M} = \varepsilon$. ◀

To finish the proof, observe that since $(y_n)_{n\in\mathbb{N}}$ converges, it is bounded, hence by the claim above $\lim_{n\to\infty} x_n y_n = 0$, but since $x = 0$ we have $xy = 0$, and the result follows.